VOLUME XIII SECOND SERIES NUMBER 3

THE PHYSICAL REVIEW

A JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS

CONDUCTED BY

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PUBLISHED BY
THE AMERICAN PHYSICAL SOCIETY
LANCASTER, PA., AND ITHACA, N. Y.

Entered at the post-office, at Lancaster, Pa., as second class matter

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Annual Subscription, throughout the Postal Union, $6.00

Single Copies, postpaid, $.60

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THE PHYSICAL REVIEW, ITHACA, N. Y.

Second Series *March, 1919* *Vol. XIII., No. 3*

THE

PHYSICAL REVIEW.

THE STRUCTURE OF RADIOACTIVE ELEMENTS.

By Ingo W. D. Hackh.

THE progress of knowledge in chemical and physical sciences has come mainly through the interpretation of experimental facts. Many phenomena of modern physical research can only be explained by theories of the structure of atoms. Thus a branch of "sub-atomic" physics and chemistry has been created. The majority of papers dealing with the structure of the atom deal with the elements of lighter atomic weight, while the elements of highest atomic weight which exhibit the phenomena of radioactivity seem to be neglected. Nevertheless these later elements offer a more substantial basis for theories concerning the constitution of atoms, because, by their disintegration some of the building stones of the atoms are revealed.

In further developing the periodic system of the chemical elements[1] the following scheme was devised for the series of the radioactive elements, by which the change in their relative electromotive force and valency during the different stages of disintegration is brought in harmony with the total charge of the respective substance. This scheme not only offers proof of the completeness of the uranium-radium series, but also points to a theory of "metastasic" electrons and incidentally establishes temporarily structure formula and logical symbols for the radioactive substances.

According to the current interpretation of disintegration the radioactive elements exhibiting beta-radiation, discharge an electron, and become relatively more negative by an increase in valency and corresponding change of their position in the periodic system.

Thus, *e. g.*, tetravalent uranium x', an isotope of thorium in group 4, loses an electron and becomes pentavalent uranium x'' or brevium[2] occupying a position in group 5, which is thus more electro-negative. This very short lived substance loses another electron and is transformed

[1] J. Am. Chem. Soc., 40, 1023, 1918. Am. J. of Science, 46, p. 481, 1918.

[2] Fajans, Phys. Z., 14, 877, 1913. See footnote, Ber., 46, 3492, 1913.

into the more stable hexavalent uranium 2 of group 6. Therefore with each loss of an electron (or negative charge) the valency has increased and the relative electromotive force of the substance has become more electro-negative. According to the conception of the conservation of energy the *total* negative charge of uranium 2 must be less than the total negative charge of uranium x'' or uranium x', although the electromotive force of uranium 2 is more electro-negative than its two predecessors.

TABLE I.

Structure of the Members of the Uranium-Radium Series. (Fig. 1.)

	Atomic Number, *AN.*		Helium Nuclei, *He.*	Electrons.			Structure Formula.
				Unstable Shell. E_i.	Valency Shell, E_v.	Total, E_m.	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
A	92	Uranium 1 . .	$8 + x$	$8 + y$	6	$14 + y$	$u\text{-}He_8E_{8+6}$
B	90	Uranium X^1 .	$7 + x$	$10 + y$	4	$14 + y$	$u\text{-}He_7E_{10+4}$
C	91	Brevium	$7 + x$	$8 + y$	5	$13 + y$	$u.He_7E_{8+5}$
D	92	Uranium 2 . .	$7 + x$	$6 + y$	6	$12 + y$	$u.He_7E_{6+6}$
E	90	Ionium	$6 + x$	$8 + y$	4	$12 + y$	$u.He_6E_{8+4}$
F	88	Radium	$5 + x$	$10 + y$	2	$12 + y$	$u.He_5E_{10+2}$
G	86	Niton	$4 + x$	$12 + y$	0	$12 + y$	$u.He_4E_{12+0}$
H	84	Radium A . . .	$3 + x$	$6 + y$	6	$12 + y$	$u.He_3E_{6+6}$
I	82	Radium B . . .	$2 + x$	$8 + y$	4	$12 + y$	$u.He_2E_{8+4}$
J	83	Radium C . . .	$2 + x$	$6 + y$	5	$11 + y$	$u.He_2E_{6+5}$
J^1	81	Radium C^{11} .	$1 + x$	$8 + y$	3	$11 + y$	$u.He_1E_{8+3}$
K	84	Radium C^1 .	$2 + x$	$4 + y$	6	$10 + y$	$u.He_2E_{4+6}$
L	82	Radium D. .	$1 + x$	$6 + y$	4	$10 + y$	$u.He_1E_{6+4}$
M	83	Radium E . . .	$1 + x$	$4 + y$	5	$9 + y$	$u.He_1E_{4+5}$
N	84	Radium F . . .	$1 + x$	$2 + y$	6	$8 + y$	$u.He_1E_{2+6}$
O	82	Radio-lead . .	x	$4 + y$	4	$8 + y$	$u\ E_{4+4}$

Assuming that the disintegration manifesting itself as alpha-radiation is the result of a *single* atom emitting a *single* helium-nucleus (which later becomes a helium-atom) and beta-radiation the result of a *single* atom discharging a *single* electron, it becomes evident that each step in disintegration must have been accompanied simultaneously by an intra-atomic shift of electrons between the outer valency ring and the interior part of the atom. This change in the position of "metastasic" electrons can occur according to the two types of disintegration, in two directions, namely; (1) in α-radiation two valency electrons move from the outer shell to the unstable inner ring and take the place of the emitted helium nucleus, decreasing the valency thereby by two and the relative electromotive force of the resulting substance will be less negative; (2) in β-

radiation there is in addition to the emission of an electron the transfer of another electron from the unstable inner ring to the outer valency ring, thus increasing the valency and making the electromotive force more negative.

There are in the uranium-radium series 16 members, eight of which exhibit alpha-radiations, six show beta-radiations and two final end products exhibit apparently no radiation. Thus the difference from uranium 1 to radio-lead consists of eight helium-nuclei and six electrons. This difference between the structure of U and Pb has been used in the calculation of the number of helium-nuclei and electrons for the different members of the series as shown in Table I. Beginning with uranium 1 there are in the unstable ring eight helium-nuclei and eight electrons, while the valency ring contains six valency electrons. If now one helium-nucleus is thrown off, two of the valency electrons take temporarily the place of the helium-nucleus, the unstable ring of the new metabolon or atom consists of seven helium-nuclei and ten electrons, while four valency electrons are left in the outer ring (uranium x'). This new system of seven He and ten E is very unstable and in the next step of disintegration one electron is thrown off entirely, while one electron

TABLE II.

Structure of the Members of the Thorium Series.

Atomic Number, *AN.*		Helium Nuclei, *He.*	Electrons.			Structure Formula.
			Unstable Shell, E_i.	Valency Shell, E_v.	Total, E_m.	
(2)	(3)	(4)	(5)	(6)	(7)	(8)
90	Thorium.....	$6 + x^1$	$8 + y^1$	4	$12 + y^1$	$t.He_6E_{8+4}$
88	Mesothorium 1...	$5 + x^1$	$10 + y^1$	2	$12 + y^1$	$t.He_5E_{10+2}$
89	Mesothorium 2...	$5 + x^1$	$8 + y^1$	3	$11 + y^1$	$t.He_5E_{8+3}$
90	Radio-thorium . .	$5 + x^1$	$6 + y^1$	4	$10 + y^1$	$t.He_5E_{6+4}$
88	Thorium X.....	$4 + x^1$	$8 + y^1$	2	$10 + y^1$	$t.He_4E_{8+2}$
86	Thorium-emana'n	$3 + x^1$	$10 + y^1$	0	$10 + y^1$	$t.He_3E_{10+0}$
84	Thorium A.....	$2 + x^1$	$4 + y^1$	6	$10 + y^1$	$t.He_2E_{4+6}$
82	Thorium B.....	$1 + x^1$	$6 + y^1$	4	$10 + y^1$	$t.He_1E_{6+4}$
83	Thorium C.....	$1 + x^1$	$4 + y^1$	5	$9 + y^1$	$t.He_1E_{4+5}$
81	Thorium D.....	x^1	$6 + y^1$	3	$9 + y^1$	$t.E_{6+3}$
84	Thorium C^1.....	$1 + x^1$	$2 + y^1$	6	$8 + y^1$	$t.He_1E_{2+6}$
82	Thorium-lead....	x^1	$4 + y^1$	4	$8 + y^1$	$t.E_{4+4}$

shifts to the outer ring, leaving in the unstable ring the seven helium nuclei and eight electrons (Ux″ or brevium), and increasing thereby the valency electrons from four to five.

In Fig. 1 this idea is graphically illustrated and the number of helium

nuclei and electrons of each member is shown. From these data are gotten the temporary structure formula shown in column 8 of Table I. In a similar way the results for the members of the thorium series are given in Table II. The sole difference between the two types of structure formula is in the value u and t. The symbol u indicates "lead from uranium," that is a nucleus with atomic weight 206.4 containing x helium nuclei and y electrons, while t indicates "lead from thorium" with atomic weight 208.2, which consists of x' helium nuclei and y'

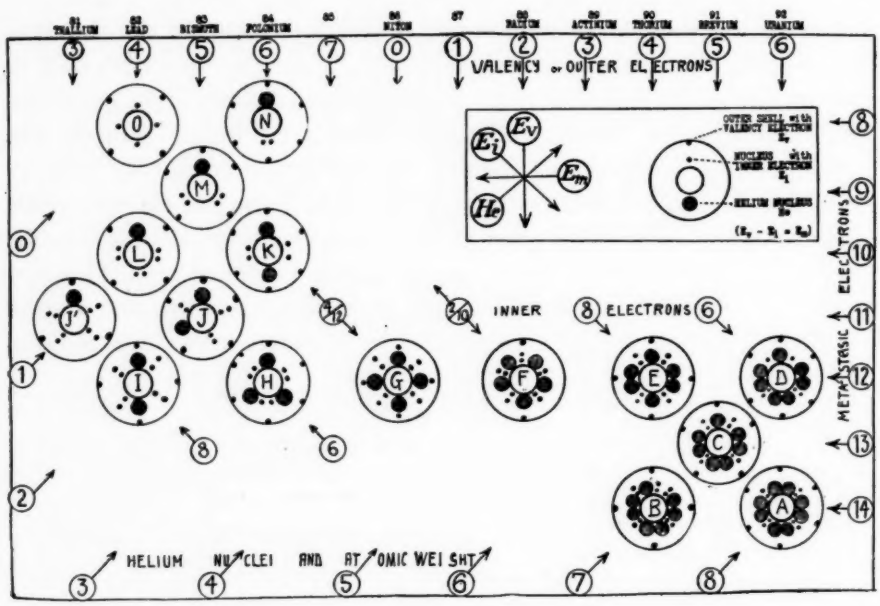


Fig. 1.

Diagram indicating the structure of the members of the uranium-radium series. The diagram is thought to be attached to the periodic system and the position of the radioactive substance in this scheme determines its structure. In any of the four axis the number of helium nuclei and electrons is the same for different members, thus the members which are located on the same southwest to northeast axis have the same atomic weight and the same number of helium nuclei (He), corresponding to the data of column 4 in Table I.; northwest to southeast axis have the same number of inner electrons (E_i), column 5 of Table I.; north to south axis belong to the same group, have the same valency (E_v) and are the isotopes of the element shown on the top of the axis (column 6 of Table I.); west to east axis have the same number of metastasic electrons (E_m), column 7 of Table I.

electrons. Whether or not there is a difference in the values x and x', and y and y', can not be definitely decided at present, for the experimental difference existing between t and u may be due to a different arrangement of the same number of helium nuclei ($x = x'$) and electrons ($y = y'$).

For the present the structure formula offer certain advantages prefer-

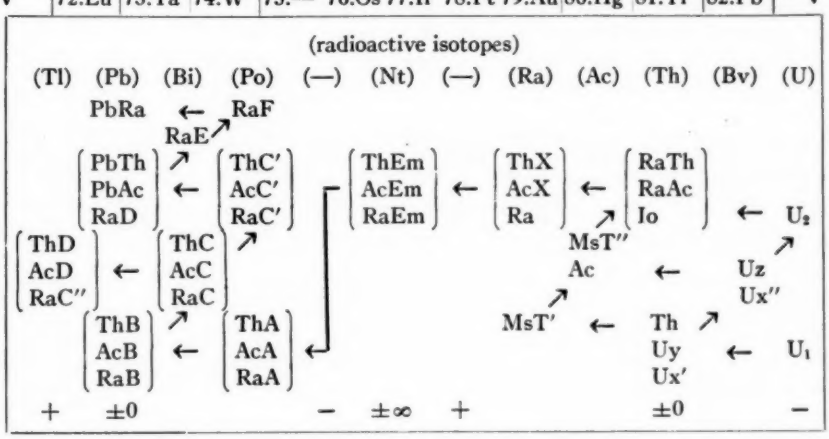
ential to the symbols employed for the radioactive substances, for they not only indicate the series (by u and t) and groups (by valency electrons), but also the number of helium nuclei and electrons. For instance the formula $u.He_2E_{6+5}$ indicates that the substance which it represents is (1) a member of the uranium series, differing from uranium I ($u.He_8E_{8+6}$) by $8 - 2$ or six helium nuclei (corresponding to six alpha-radiations), and $(8 + 6) - (6 + 5)$ or three electrons (= three beta-radiations), and is therefore the ninth member of the series; (2) it possesses an atomic weight of 214.4, which is the sum of $u = 206.4$ and $He_2 = 8.0$; (3) it contains $(6 + 5)$ or eleven metastasic electrons of which five are in the outer valency shell and it belongs therefore to group five of the periodic system, and is therefore an isotope of bismuth.

The completeness of both series from uranium or thorium to lead, be-

TABLE III.

The Periodic System and the Radioactive Isotopes.

	4	5A	6A	7A	0	1A	2A	3A	4	
		(non-metals)			(inert gases)	(light-metals)				
	±0	−	−−	−−−	±∞	+++	++	+	±0	
Vb	82.Pb	83.Bi	84.Po	85.—	86.Nt	87.—	88.Ra	89.Ac	90.Th	VI
IVb	50.Sn	51.Sb	52.Te	53.I	54.Xe	55.Cs	56.Ba	57.La	58.Ce	Va
IIIb	32.Ge	33.As	34.Se	35.Br	36.Kr	37.Rb	38.Sr	39.Y	40.Zr	IVa
IIb	14.Si	15.P	16.S	17.Cl	18.Ar	19.K	20.Ca	21.Sc	22.Ti	IIIa
Ib	6.C	7.N	8.O	9.F	10.Ne	11.Na	12.Mg	13.Al	14.Si	IIa
0		1.H			2.He	3.Li	4.Be	5.B	6.C	Ia
		(heavy metals)								
III′	22.Ti	23.V	24.Cr	25.Mn	26.Fe 27.Co 28.Ni 29.Cu		30.Zn	31.Ga	32.Ge	III′
IV′	40.Zr	41.Cb	42.Mo	43. —	44.Ru 45.Rh 46.Pd 47.Ag		48.Cd	49.In	50.Sn	IV′
V″	58.Ce	Pr	Nd	—	Sa Eu Gd Tb Dy Ho Er Tm		Db	Yb	72.Lu	V″
V′	72.Lu	73.Ta	74.W	75.—	76.Os 77.Ir 78.Pt 79.Au		80.Hg	81.Tl	82.Pb	V′



comes evident by the periodic system because all places are occupied according to the rule that the position is changed two places to the left by alpha-radiation and one place to the right by beta-radiation. The deviation of the actinium-series is still doubtful and the structure formula of its members have therefore not been calculated.

The relationship of the radioactive elements to the other elements is shown in Table III. In this table the radioactive elements are appended to the new periodic table, the isotopes being placed vertically under each other. It also shows that the disintegration of all three series runs parallel, beginning with RaTh, RaAc and Io.

CONCLUSIONS.

1. The phenomena of disintegration enables the establishment of a partial atom-structure formula for each radioactive element. These atom-structure formulæ indicate the atomic weight, the valency or place in the periodic system and the respective position in the uranium-radium or thorium-series.

2. A theory of metastasic electrons has been developed by which (*a*) alpha-radiation or the emission of a helium-nucleus causes the change of position of two electrons from the valency ring to the interior unstable ring, while (*b*) beta-radiation or the emission of an electron causes the shift of another electron from the inner or unstable ring to the outer or valency ring.

3. This theory supports the view that the seat of disintegration is located in the next inner shell, and not in the outer or valency shell, and is caused by the shifting of two electrons.

4. The periodic table offers indications that the uranium-radium series and the thorium series down to lead are completely known and that no more members of these two series can be expected.

ON THE VIBRATIONS OF ELASTIC SHELLS PARTLY FILLED WITH LIQUID.

BY SUDHANSUKUMAR BANERJI.

I. INTRODUCTION.

THE problem considered in this paper is chiefly of acoustical interest in relation to the theory of "musical glasses." This class of instrument consists of a series of thin-walled elastic shells whose gravest modes of vibration are tuned to form a musical seal by partially filling them with a liquid and are excited either by striking or by tangential friction on the ruins. The principal features of interest requiring elucidation are (1) the dependence of the pitch of the vibration upon the quantity of liquid contained in the vessel and (2) the mode of vibration of the liquid itself. These features are discussed in this paper for the three cases in which the elastic shell is respectively (1) a hemispherical shell, (2) a cylindrical vessel with a flat bottom and (3) a conical cup, these forms approximating more or less closely to those used in practice. The analytical expressions show that the motion of the liquid is very marked near the margin of the vessel and is almost imperceptible near the center and also at some depth inside the liquid. This feature becomes more and more marked in the case of the higher modes of vibration of the vessel. Numerical results have also been obtained and tabulated showing the theoretical relation between the quantity of liquid contained in the vessel and the vibration frequency. These show that the rapidity with which the frequency falls on addition of liquid is greatest when the vessel is nearly full, this being specially noticeable in the case of the higher modes of vibration.

The general principle of the analytical method used is similar to that adopted by Lord Rayleigh[1] in treating the two-dimensional case of a long cylinder completely filled with liquid which was studied by Auerbach.[2] This case has also been recently discussed by Nikoloi.[3] The lowering of

[1] Lord Rayleigh, Phil. Mag., XV., pp. 385–389 (1883). Scientific Papers, Vol. 2, pp. 208–211.

[2] Auerbach, Wied. Ann., 3, p. 157, 1878, and also Wied. Ann., 17, p. 964, 1882. Reference may also be made to the papers by Montigny, Bull. del' Acad. de Belg., [2], 50, 159, 1880, and by Koläcek, Wied. Ann., 7, 23, 1879, and also Sitz. math. naturw. d. Wien, 87, Abath. 2, 1883.

[3] Nilokoi, Journ. Russk. Fisik Chimicesk, 41, 5, pp. 214–227, 1909.

the pitch produced by the liquid is of course due to the added inertia exactly as in the related case of the vibrations of a bar or a string immersed in a liquid which have been studied by Northway,[1] Mackenzie and Kalähue.[2]

Musical glasses are sometimes excited by rotating them about a fixed vertical axis, the tangential friction being produced by a rubber kept in a fixed position. No attempt is made in this paper to consider this somewhat complicated case,[3] which I hope to be able to deal with on a future occasion.

2. HEMISPHERICAL CUPS.

The force which a thin sheet of matter subjected to stress opposes to extension is very great in comparison with that which it opposes to bending. From this Lord Rayleigh concluded that the middle surface of a vibrating shell remains unstretched and proposed a theory[4] of flexural vibrations of curved plates and shells in accordance with this condition. As the direct application of the Kirchhoff-Gehring method led to equations of motion and boundary conditions which were difficult to reconcile with Lord Rayleigh's theory, his theory gave rise to much discussion. Later investigations have, however, shown that any extension that may occur must be limited to a region of infinitely small area near the edge of the shell and that the greater part of the shell vibrates according to Lord Rayleigh's type.

Let the radius of the hemisphere be equal to a. Let a point whose natural coördinates are a, θ, ϕ be displaced to the position $a + u$, $\theta + v$, $\phi + w$, where u, v, w are to be treated as small.

From the condition of inextension

$$(\delta s)^2 = a^2(\delta\theta)^2 + a^2 \sin^2\theta(\delta\phi)^2$$

$$= (a + u)^2(\delta\theta + \delta v)^2 + (\delta u)^2 + (a + u)^2 \sin^2(\theta + v)\,(\delta\phi + \delta w)^2, \quad (1)$$

Lord Rayleigh obtains the three differential equations

$$\frac{\partial v}{\partial\theta} + \frac{u}{a} = 0,$$

[1] Northway and Mackenzie, PHYS. REV., 13, pp. 145–164, 1901.

[2] Kalähne, Ann. d. Physik, 46, 1, pp. 1–38, 1914.

[3] Reference may be made in this connection to papers by Prof. Love on "The Free and Forced Vibrations of an Elastic Spherical Shell Containing a Given Mass of Liquid," Proc. Lond. Math. Soc., Vol. XIX., where he has studied the case of a rotating spherical shell completely full of liquid, and by Prof. Bryan on "The Beats in the Vibrations of a Revolving Cylinder or Bell," Proc. Comb. Phil. Soc., Vol. VII., 1892.

[4] Lord Rayleigh, Proc. Lond. Math. Soc., Vol. XIII., p. 4, 1881. See also Proc. Roy. Soc., Vol. 45, pp. 45 and 443, 1881, Theory of Sound, Vol. I., Chap. XV, and Love's Elasticity, Chap. XXXIII.

$$\frac{\partial v}{\partial \phi} + \sin^2\theta \frac{\partial w}{\partial \theta} = 0,$$

$$\frac{u}{a} + \cot\theta \cdot v + \frac{\partial w}{\partial \phi} = 0,$$

which can be integrated in the forms

$$\frac{u}{a} = \begin{matrix}\sin \\ -\cos\end{matrix} m\phi[A_m(m + \cos\theta)\tan^m \tfrac{1}{2}\theta - B_m(m - \cos\theta)\cot^m \tfrac{1}{2}\theta],$$

$$\frac{v}{\sin\theta} = \begin{matrix}-\sin \\ \cos\end{matrix} m\phi[A_m \tan^m \tfrac{1}{2}\theta - B_m \cot^m \tfrac{1}{2}\theta],$$

$$w = \begin{matrix}\cos \\ \sin\end{matrix} m\phi[A_m \tan^m \tfrac{1}{2}\theta + B_m \cot^m \tfrac{1}{2}\theta],$$

A_m and B_m being arbitrary constants. These equations determine the character of the displacement of a point in the middle surface.

Since the pole $\theta = 0$ is included the constant B_m must be considered to vanish and the type of vibrations in a principal mode is expressed by the equations

$$\begin{aligned} u &= A_m\alpha(m + \cos\theta)\tan^m \tfrac{1}{2}\theta \sin m\phi, \\ v &= -A_m \sin\theta \tan^m \tfrac{1}{2}\theta \sin m\phi, \\ w &= A_m \tan^m \tfrac{1}{2}\theta \cos m\phi, \end{aligned} \tag{4}$$

in which A_m is proportional to a simple harmonic function of the time.

The potential energy of bending of the vibrating shell is given by

$$\begin{aligned} V &= \frac{8}{3}\pi\mu\frac{\tau^3}{a^2}m^2(m^2 - 1)^2 A_m^2 \int_0^{\pi/2} \tan^{2m}\frac{\theta}{2}\frac{d\theta}{\sin^3\theta} \\ &= \frac{2}{3}\pi\mu\frac{\tau^3}{a^2}(m^3 - m)(2m^2 - 1)A_m^2, \end{aligned} \tag{5}$$

where τ = thickness of the shell and μ = rigidity.

The kinetic energy T is given by the expression

$$\begin{aligned} T &= \frac{1}{2}\pi\sigma\alpha^4\tau\left(\frac{dA_m}{dt}\right)^2 \int_0^{\pi/2} \sin\theta\{2\sin^2\theta + (\cos\theta + m)^2\}\tan^{2m}\tfrac{1}{2}\theta d\theta \\ &= \frac{1}{2}\pi\sigma\alpha^4\tau\left(\frac{dA_m}{dt}\right)^2 \int_1^2 \frac{(2 - x)^m}{2^m}[(m - 1)^2 + 2(m + 1)x - x^2]dx \\ &= \frac{1}{2}\pi\sigma\alpha^4\tau f(m)\left(\frac{dA_m}{dt}\right)^2, \end{aligned} \tag{6}$$

where σ represents the density of the shell, and

$$f(m) = \int_1^2 \frac{(2 - x)^m}{x^m}[(m - 1)^2 + 2(m + 1)x - x^2]dx, \tag{7}$$

which can be evaluated for any integral value of m.

Since the types of vibrations of the shell are entirely determined by the geometry of the middle surface of the shell, it is obvious that the types can under no circumstances be altered by the presence of the liquid in the shell. The liquid gives rise to a surface traction and affects only the arbitrary constant A_m, that is to say, the amplitude and the frequency of vibration of the shell.

The motion of the liquid will depend upon a velocity potential which satisfies the equation

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}+\frac{\operatorname{cosec}^2\theta}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}=0. \tag{8}$$

A solution of this differential equation which will correspond to the type of vibration of the shell can be obtained by assuming Φ to be of the form

$$\Phi=\left(C_m r+\frac{D_m}{r^2}\right)\sin m\phi\cdot\Delta_\theta,$$

where Δ_θ is a function of θ only. Substituting in the differential equation we find that Δ_θ satisfies the equation

$$\frac{d^2\Delta_\theta}{d\theta^2}+\cot\theta\frac{d\Delta_\theta}{d\theta}+(2-m^2\operatorname{cosec}^2\theta)\Delta_\theta=0.$$

The general solution of this differential equation is

$$\Delta_\theta=E_m\tan^m\tfrac{1}{2}\theta(m+\cos\theta)+F_m\cot^m\tfrac{1}{2}\theta(m-\cos\theta).$$

Neglecting solutions of the type $\cot^m\frac{1}{2}\theta(m-\cos\theta)$, we see that Φ is of the form

$$\Phi=\left(C_m r+\frac{D_m}{r^2}\right)\sin m\phi\tan^m\tfrac{1}{2}\theta(m+\cos\theta), \tag{9}$$

where C_m and D_m are two arbitrary constants. Let us first take

$$\Phi=C_m\frac{r}{a}\tan^m\tfrac{1}{2}\theta(m+\cos\theta)\sin m\phi\cos pt. \tag{10}$$

The relation between C_m and A_m of (4) is readily found by equating the value of $\partial\Phi/\partial r$, when $r=a$, to $\partial u/\partial t$, both of which represent the normal velocity at the circumference. We get

$$C_m\cos pt=a^2\frac{dA_m}{dt}. \tag{11}$$

The expression (10) determines the principal mode of vibration of the liquid. The simple character of the fluid motion as determined by this expression will however be a little disturbed on account of the existence of a free surface and we shall have to add a small correction to this

expression. The condition to be satisfied at the free surface is

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0, \text{ when } z = h,$$

where h denotes the depth of the liquid surface below the center of the hemisphere. We shall neglect the force of gravity, inasmuch as the period of free waves of length comparable with the diameter of the shell is much greater than that of the actual motion. The condition to be satisfied at the free surface then becomes simply

$$\Phi = 0, \quad \text{when } z = h.$$

Hence we must have

$$\Phi = C_m\frac{r}{a}\tan^m \tfrac{1}{2}\theta(m + \cos\theta)\sin m\phi\cos pt + f(r, \theta, \phi)\cos pt, \quad (12)$$

where $f(r, \theta, \phi)$ is a solution of $\Delta^2\Phi = 0$ and is such that its differential coefficient with respect to r vanishes on the spherical boundary and it has the value

$$-C_m\frac{h\sec\theta}{a}\tan^m \tfrac{1}{2}\theta(m + \cos\theta)\sin m\phi \quad (13)$$

on the free surface.

In the particular case, when the shell is completely full of liquid, the differential coefficient of $f(r,\theta,\phi)$ with respect to r vanishes on the spherical surface and $f(r, \theta, \phi)$ has the value

$$-mC_m\frac{r}{a}\sin m\phi \quad (14)$$

on the surface defined by $\theta = \pi/2$.

For the determination of the function $f(r, \theta, \phi)$, spherical harmonics of the complex degree $-\frac{1}{2} + p\sqrt{-1}$ are extremely suitable. The properties of these harmonics and their applications to some physical problems have been investigated by Hobson.[1] Solutions of Laplace's equation of the form

$$\frac{1}{\sqrt{r}}\sin(p\log Ar)\begin{matrix}\sin\\ \cos\end{matrix} m\phi K_p^m(\cos\theta),$$

where $K_p^m(\cos\theta)$ is a harmonic of degree $-\frac{1}{2} + p\sqrt{-1}$ and order m, and is defined by the hypergeometric series

$$K_p^m(\cos\theta) = F(m + \tfrac{1}{2} + pi, m + \tfrac{1}{2} - pi, m + 1, \sin^2\tfrac{1}{2}\theta),$$

are suitable for our present purpose. These solutions are finite and continuous for all points in the space inside the hemispherical shell (except

[1] Hobson, "On a Class of Spherical Harmonics of Complex Degree with Application to Physical Problems," Trans. Camb. Phil. Soc., Vol. 14, pp. 212–236, 1889.

infinitely near the origin which may be supposed excluded by surrounding it by an infinitely small sphere).

Let us assume

$$f(r, \theta, \phi) = \Sigma_p B_p \sqrt{\frac{a}{r}} \sin\left(p \log \frac{r}{h}\right) \frac{K_p^m(\cos\theta)}{K_p^m(\cos\alpha)} \sin m\phi, \qquad (15)$$

where $h/a = \cos\alpha$. Then

$$\frac{\partial}{\partial r} f(r, \theta, \phi) = 0,$$

when $r = a$, if the p's are the roots of the equation

$$\tan(p \log a/h) - 2p = 0,$$

and the summation in the above series extends over all the roots of this equation.

The values of the constant B_p have to be obtained from the equation

$$-Cm \frac{\cos\alpha}{\cos\theta} \tan^m \tfrac{1}{2}\theta (m + \cos\theta) = \Sigma_p B_p \sqrt{\frac{a}{h}} \frac{\sin[p \log(\sec\theta)]}{\sqrt{\sec\theta}} \frac{K_p^m(\cos\theta)}{K_p^m(\cos\alpha)},$$

which must be satisfied for all values of θ between the limits $0 < \theta < \alpha$. Approximate values of the constants B_p's can be easily obtained from this equation. In the particular case when the shell is completely full of liquid, the values of the constants B_p's can be obtained in a very simple form. Since the origin is a singular point, we exclude the point by surrounding it with a small sphere of radius ϵ, and assume that $f(r, \theta, \phi)$ vanishes on the surface of this sphere. Since in this case $\alpha = \pi/2$, we can assume

$$f(r, \theta, \phi) = \Sigma_p B_p \sqrt{\frac{a}{r}} \sin\left(p \log \frac{r}{\epsilon}\right) \frac{K_p^m(\cos\theta)}{K_p^m(0)} \sin m\phi,$$

where

$$K_p^m(0) = \frac{\sqrt{\pi}\, \Pi(m)}{\Pi(\frac{1}{2}m - \frac{1}{4} + \frac{1}{2}pi)\,\Pi(\frac{1}{2}m - \frac{1}{4} - \frac{1}{2}pi)}$$

$$= \frac{\sqrt{\pi}\,\Pi(m)}{\left\{\frac{(2m-1)^2 + p^2}{2^2}\right\}\left\{\frac{(2m-3)^2 + p^2}{2^2}\right\} \cdots}$$

and the summation extends for all values of p which are the roots of the equation

$$\frac{d}{da}\left[\frac{\sin[p \log(a/\epsilon)]}{\sqrt{a}}\right] = 0,$$

that is to say, the equation

$$\tan\left(p \log \frac{a}{\epsilon}\right) - 2p = 0. \qquad (16)$$

The constants B_p's have to be determined by the condition that $f(r, \theta, \phi)$ must have the value $-mC_m(r/a)\sin m\phi$ on the free surface which is given by $\theta = \pi/2$. Hence B_p's are given by

$$-mC_m\frac{r}{a} = \Sigma_p B_p\sqrt{\frac{a}{r}}\sin\left(p\log\frac{r}{\epsilon}\right).$$

Putting $r = \epsilon\cdot e^{\lambda}$, it is easy to see that

$$B_p = -\frac{2(p^2+\frac{1}{4})\cdot mC_m}{p^2\log(a/\epsilon)+\frac{1}{2}[\frac{1}{2}\log(a/\epsilon)-1]}\left(\frac{\epsilon}{a}\right)^{3/2}\int_0^{\log a/\epsilon} e^{3/2\lambda}\sin p\lambda d\lambda$$

$$= -\frac{8(4p^2+1)\cdot mC_m}{4p^2\log(a/\epsilon)+[\log(a/\epsilon)-2]}\left[\frac{\sin[p\log(a/\epsilon)]+p(\epsilon/a)^{3/2}}{9+4p^2}\right]. \quad (17)$$

To obtain an idea of the magnitude of the constant B_p, we shall obtain its value when a/ϵ is a very large quantity. It is easy to see by the method of successive approximation that the roots of the equation (16) are given by

$$p\log\frac{a}{\epsilon} = X - \frac{1}{2X}\log\frac{a}{\epsilon} - \left(\log\frac{a}{\epsilon}\right)^2\left[\frac{1}{4} - \frac{1}{2^4}\log\frac{a}{\epsilon}\right]\frac{1}{X^3}$$

$$- \left(\log\frac{a}{\epsilon}\right)^3\left[\frac{1}{4} - \frac{1}{12}\log\frac{a}{\epsilon} + \frac{1}{160}\left(\log\frac{a}{\epsilon}\right)^2\right]\frac{1}{X^5}$$

$$- \text{etc.},$$

where $X = (s+\frac{1}{2})\pi$, s being any integer. Now, if we take $a/\epsilon = 10^5$, the roots are successively the following:

$$p_1 = \cdot 321,\ p_2 = \cdot 628,\ p_3 = \cdot 960,\ p_4 = 1.205, \text{ etc.}$$

Hence we easily find that the constants B_{p1}, B_{p2}, B_{p3}, etc., have approximately the values $B_{p_1} = -\cdot 09\ C_m$, $B_{p_2} = -\cdot 08\ C_m$, $B_{p_3} = -\cdot 06\ C_m$, etc., from which we infer that the surface correction $f(r, \theta, \phi)$ is a small one. The principal mode of vibration of the liquid is therefore expressed by

$$\Phi = C_m\frac{r}{a}\tan^m\tfrac{1}{2}\theta(m+\cos\theta)\sin m\phi\cos pt. \quad (18)$$

If q represent the velocity of the liquid as given by this expression, we have

$$q^2 = C_m^2\frac{1}{a^2}\tan^{2m-2}\tfrac{1}{2}\theta[(m+\cos\theta)^2(\sin^2 m\phi\tan^2\tfrac{1}{2}\theta+\tfrac{1}{4}m^2\cos^2 m\phi\sec^2\tfrac{1}{2}\theta)$$

$$+ \{\tfrac{1}{2}m(m+\cos\theta)\sec^2\tfrac{1}{2}\theta - \tan\tfrac{1}{2}\theta\sin\theta\}^2\sin^2 m\phi]\cos^2 pt. \quad (19)$$

Since q is independent of r, the velocity of the liquid at any point in a given radius vector is constant. We see that the velocity varies as $\tan^{m-1}\frac{1}{2}\theta$. Hence if we move along any given meridian, the velocity increases from a zero value at the pole at first very slowly then rather abruptly to a

large value at the surface, the abruptness of rise being greater the larger the quantity m, that is to say, the higher the mode of vibration of the liquid. Since the velocity of the liquid is constant along any given radius vector, we see that if we consider the motion of the liquid on the surface of a cone of semi-vertical angle θ, and trace the motion of the liquid as a whole as θ increases, the velocity remains small as θ increases and assumes a large value only at or near the surface. It is obvious therefore that in every case when the cup is not quite filled to the brim, the velocity of the liquid has a very large value near the margin of the vessel and is almost imperceptible near the center and at some depth in the liquid. In the particular case, when the shell is almost filled to the brim, the velocity of the liquid as given by this expression at a point which is near the center and also near the free surface is not small. But in this case the free surface correction $f(r, \theta, \phi)$ to the expression for the velocity potential becomes of some importance and has a sign opposite to it. Consequently the velocity of the liquid near the center always remains very small. These indications of theory are all confirmed by experiment.

To calculate the kinetic energy of the liquid, we have to integrate $\Phi \times \partial\Phi/\partial n$ over the whole boundary of the fluid. At the free surface $\Phi = 0$. We have therefore only to consider the spherical surface.

Therefore

$$
\begin{aligned}
T &= \tfrac{1}{2}\rho \int\int \Phi \frac{\partial \Phi}{\partial n} dS \\
&= \tfrac{1}{2}\alpha\rho \cos^2 pt \int_0^{2\pi} \int_0^{\alpha} [C_m \sin m\phi \tan^m \tfrac{1}{2}\theta(m + \cos\theta) + f(r, \theta, \phi)] \\
&\times C_m \sin m\phi \tan^m \tfrac{1}{2}\theta(m + \cos\theta) \sin\theta d\theta d\phi \\
&= \frac{\pi}{2} a\rho \cos^2 pt\, C_m{}^2 \int_0^{\alpha} \tan^{2m} \tfrac{1}{2}\theta(m + \cos\theta)^2 \sin\theta d\theta \qquad (20) \\
&+ \frac{\pi}{2} a\rho \cos^2 pt\, C_m \sum_p \beta_p \frac{\sin [p \log (a/h)]}{K_p{}^m (\cos\alpha)} \\
&\qquad \times \int_0^{\alpha} \tan^m \tfrac{1}{2}\theta(m + \cos\theta) K_p{}^m (\cos\theta) \sin\theta d\theta,
\end{aligned}
$$

ρ being the density of the liquid.

Since the liquid is supposed to be incompressible, the potential energy is zero.

The sum of the kinetic and potential energies of the solid and liquid together must be independent of the time. Thus we get

$$
\left[a^5\rho \int_0^{\alpha} \tan^{2m} \tfrac{1}{2}\theta(m + \cos\theta)^2 \sin\theta d\theta + \alpha^5\rho K + \alpha^4 \tau\sigma f(m) \right] \frac{d^2 A_m}{dt^2}
$$
$$
+ \frac{4}{3}\mu \frac{\tau^3}{\alpha^2}(m^3 - m)(2m^2 - 1) A_m = 0,
$$

where

$$K = \sum_p \frac{B_p}{C_m} \frac{\sin [p \log (a/h)]}{K_p^m (\cos \alpha)} \int_0^\alpha \tan^m \tfrac{1}{2}\theta (m + \cos \theta) K_p^m (\cos \theta) \sin \theta d\theta$$

$$= \sum_p \frac{B_p}{C_m} \frac{4 \sin^2 \alpha}{9 + 4p^2} \cdot \frac{\sin [p \log (a/h)]}{K_p^m (\cos \alpha)} \Big[K_p^m (\cos \alpha) \frac{d}{d \cos \alpha} \{ \tan^m \tfrac{1}{2}\alpha (m + \cos \alpha) \}$$

$$- \tan^m \tfrac{1}{2}\alpha (m + \cos \alpha) \frac{d}{d \cos \alpha} K_p^m (\cos \alpha) \Big].$$

If we put

$$F(\alpha, m) = \int_0^\alpha \tan^{2m} \tfrac{1}{2}\theta (m + \cos \theta)^2 \sin \theta d\theta$$

$$= \int_{1+\cos\alpha}^2 \left(\frac{2 - x}{x} \right)^m (m - 1 + x)^2 dx,$$

and if A_m varies as $\cos (pt + \epsilon)$, we get

$$[a\rho F(\alpha, m) + \tau\sigma f(m) + \alpha\rho K] p^2 = \frac{4}{3} \frac{\mu}{\alpha^3} \left(\frac{\tau}{\alpha} \right)^3 (m^3 - m)(2m^2 - 1). \quad (21)$$

This equation gives the frequency of vibration of the shell with different quantities of liquid.

The fall of pitch for the three gravest tones given by $m = 2$, $m = 3$ and $m = 4$ for a brass hemispherical shell 10 cm. in radius, 2 mm. in thickness and of density 8.6 with different quantities of liquid are shown in Table I. In Fig. 1, the frequencies have been plotted against the quantity of water in the vessel for these three modes of vibrations.

TABLE I.

α.	Quantity of Water in the Shell.	$m = 2$.			$m = 3$.			$m = 4$.		
		$F(\alpha, m)$.	$f(m)$.	$\times \frac{p}{\text{Const.}}$	$F(\alpha, m)$.	$f(m)$.	$\times \frac{p}{\text{Const.}}$	$F(\alpha, m)$.	$f(m)$.	$\times \frac{p}{\text{Const.}}$
90°	$\pi a^3 \times .667$	1.114	1.53	1.80	1.580	1.88	4.63	2.030	2.296	8.76
80°	$\pi a^3 \times .494$	.570	"	2.24	.641	"	6.50	.479	"	14.51
70°	$\pi a^3 \times 338$	.291	"	2.75	.125	"	9.53	.097	"	19.42
60°	$\pi a^3 \times 208$	.123	"	3.29	.032	"	10.57	.009	"	21.45
50°	$\pi a^3 \times .133$	.058	"	3.61	.008	"	10.95	.002	"	21.64
40°	$\pi a^3 \times .034$	.026	"	3.81	.003	"	11.14	.000	"	21.68
30°	$\pi a^3 \times .014$	.015	"	3.88	.001	"	11.22	.000	"	21.69
20°	$\pi a^3 \times .003$	.013	"	3.90	.000	"	11.23	.000	"	21.69
10°	$\pi a^3 \times .001$	.011	"	3.91	.000	"	11.23	.000	"	21.69
0°	0	0	"	3.93	0	"	11.24	0	"	21.69

The frequencies of a brass hemispherical shell of about the same radius and thickness loaded with different quantities of water have also been

determined experimentally by a photographic method. The results showed a general agreement with the calculated values. As a shell of uniform thickness and of uniform elastic properties throughout could not be procured, and the one that was used was very much deficient in these respects, the slight discrepancy that was noticed between the calculated and the observed values of the frequency, was probably due to these defects.

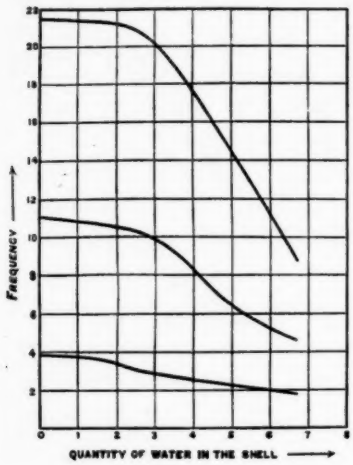


Fig. 1.

3. Cylindrical Cups.

The problem of the flexural vibrations of a cylindrical shell is considered in Lord Rayleigh's Theory of Sound, Vol. I., §235C. If the displacements at any point a, θ, z of the cylinder be δr, $a\delta\theta$, δz, then

$$\begin{aligned} \delta r &= -n(A_n a + B_n z)\sin n\theta, \\ a\delta\theta &= (A_n a + B_n z)\cos n\theta, \\ \delta z &= -n^{-1}B_n a \sin n\theta. \end{aligned} \tag{22}$$

Supposing now that the cup has been formed by an inextensible disk being attached to the cylinder at $z = 0$, the displacements δr, $a\delta\theta$ must vanish for that value of z. Hence $A_n = 0$, and

$$\delta r = -nB_n z \sin n\theta,\ a\delta\theta = B_n z \cos n\theta,\ \delta z = -n^{-1}B_n a \sin n\theta, \tag{23}$$

the constant B_n is proportional to a simple harmonic function of the time, say, $\cos pt$.

Since the displacements δr and $a\delta\theta$ are proportional to z and the displacement δz is independent of z, it is obvious, that when z is large the displacements δr and $a\delta\theta$ are also very large compared to δz, that is to say, near the free end of the shell, the displacement δz is negligible compared to δr or $a\delta\theta$. But at the bottom of the shell, the displacements δr and $a\delta\theta$ vanishes and δz remains constant. We conclude, therefore, from the law of continuity, that the disk at the bottom of the shell must have a small normal vibration. If w denote the normal displacement of the disk, it is well known that w satisfies the differential equation

$$\frac{\partial^2 w}{\partial t^2} + c^4\nabla^4 w = 0, \tag{24}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2} \text{ and } C^4 = \frac{E\tau^{12}}{3\sigma(1-\epsilon^2)},$$

E being Young's modulus, σ the density, τ^1 the thickness and ϵ the Poisson's ratio.

If $w\alpha \cos(pt + e)$, then the equation becomes

$$\nabla^4 w - v^4 w = 0,$$

where $v^4 = p^2/c^4$. A solution of this differential equation is known to be

$$w = C_n J_n(vr) \sin n\theta.$$

Hence we shall take

$$w = C_n J_n(vr) \sin n\theta \cos pt. \tag{25}$$

The value of the constant C_n can be obtained from the condition that w and δz must be continuous at the boundary.

This gives

$$C_n \cos pt = -\frac{B_n}{n} \frac{a}{J_n(va)}. \tag{26}$$

We can assume that $J_n(va)$ is very large and consequently that the normal vibration of the disk is very small. The potential energy of deformation for a length l of the cylinder is

$$V = \frac{4\pi\mu\tau^3}{3a}(n^2 - 1)^2 \left[\frac{\lambda + \mu}{\lambda + 2\mu} \frac{n^2 l^2}{3a^2} + 1\right] B_n^2. \tag{27}$$

The potential energy of vibration of the disk is given by

$$\frac{E\tau^{13}}{3(1 - \epsilon^2)} \int_0^a \int_0^{2\pi} \left[(\nabla^2 w)^2 - 2(1 - \epsilon)\left\{\frac{\partial^2 w}{\partial n^2} \frac{\partial^2 w}{\partial y^2}\right\} - \left(\frac{\partial^2 w}{\partial n \partial y}\right)^2\right] r d\theta dr.$$

The value of this integral can be easily obtained. But as we regard the vibration of the disk compared to that of the cylindrical surface to be very small, the value of this expression is also very small.

If the volume density be σ, we get the expression for the kinetic energy in the form

$$T = \tfrac{1}{2}\pi\sigma\tau l a[\tfrac{1}{3}l^2(1 + n^2) + n^{-2}a^2]\left(\frac{dB_n}{dt}\right)^2 + \tfrac{1}{2}\pi\sigma\tau' \frac{a^2}{n^2[J_n(va)]^2} \int_0^a [J_n(vr)]^2 r dr \left(\frac{dB_n}{dt}\right)^2. \tag{28}$$

If the cylinder contains frictionless incompressible fluid, the motion of the liquid will depend upon a velocity potential Φ which satisfies the equation $\nabla^2\Phi = 0$, or in cylindrical coördinates

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

The solution of this differential equation can be written in either of the

forms

$$\Phi = \alpha_n z r^n \sin n\theta \cos pt, \tag{29}$$

$$\Phi = \beta_n e^{-kz} J_n(kr) \sin n\theta \cos pt. \tag{30}$$

The boundary conditions to be satisfied by Φ are

$$\text{(i)} \quad \frac{\partial \Phi}{\partial r} = \frac{d\delta r}{dt}, \text{ when } r = a;$$

$$\text{(ii)} \quad \frac{\partial \Phi}{\partial z} = \frac{dw}{dt}, \text{ when } z = 0; \tag{31}$$

$$\text{(iii)} \quad \Phi = 0, \text{ at the free surface, } u, \text{ when } z = h.$$

We assume that

$$\Phi = \alpha_n z r^n \sin n\theta \cos pt + \sum_k J_n(kr)[D_k \cosh kz + E_k \sinh kz] \sin n\theta \cos pt, \tag{32}$$

where the summation extends for all values of k which are the roots of the equation

$$\frac{d}{da} J_n(ka) = 0. \tag{33}$$

We at once get by condition (*i*)

$$\alpha_n \cos pt = -\frac{1}{a^{n-1}} \frac{dB_n}{dt}.$$

The condition (*ii*) gives

$$-\left[\frac{a}{n} \frac{J_n(vr)}{J_n(va)} - \frac{r^n}{a^{n-1}}\right] \frac{1}{k} \frac{dB_n}{dt} = \sum_k E_k J_n(kr) \cos pt.$$

This equation must be satisfied for all values of r between the limits ($0 < r < a$) and will give the value of the constant E_k. Now since

$$\int_0^a r^{n+1} J_n(kr) dr = \frac{na^n}{k^2} J_n(ka),$$

$$\int_0^a J_n(kr) J_n(vr) r dr = \frac{av}{k^2 - v^2} J_n(ka) J_n'(va),$$

and

$$\int_0^a [J_n(kr)]^2 r dr = \tfrac{1}{2} a^2 \left(1 - \frac{n^2}{k^2 a^2}\right) [J_n(ka)]^2,$$

we get

$$-\frac{1}{k} \frac{dB_n}{dt} \left[\frac{a}{n J_n(va)} \int_0^a J_n(kr) J_n(vr) r dr - \frac{1}{a^{n-1}} \int_0^a r^{n+1} J_n(kr) dr\right] = E_k \cos pt \int_0^a [J_n(kr)]^2 r dr,$$

and therefore

$$E_k \cos pt = -\frac{2a}{ka} \frac{dB_n}{dt} \frac{k^2 a^2}{(k^2 a^2 - n^2) J_n(ka)} \left[\frac{v}{(k^2 - v^2) n} \frac{J_n'(va)}{J_n(va)} - \frac{na}{k^2 a^2}\right]. \tag{35}$$

The condition (iii) gives

$$\alpha_n h r^n + \sum_k (D_k \cosh kh + E_k \sinh kh)\, J_n(kr) = 0$$

for all values of r between the limits ($0 < r < a$).
Therefore we get

$$D_k \cosh kh + E_k \sinh kh = -\alpha_n \frac{2na^n h}{(k^2a^2 - n^2)J_n(ka)}. \qquad (36)$$

The equations (35) and (36) give the values of the constants D_k and E_k.

To calculate the kinetic energy we have to integrate $\Phi \times \partial\Phi/\partial n$ over the boundary of the shell. At the free surface $\Phi = 0$. We have therefore only to consider the cylindrical surface and the bottom. The expression can be written in the form

$$T = \tfrac{1}{2}\pi\rho \cos^2 pt \Bigg[n\alpha_n^2 a^{2n} \frac{h^3}{3} + n\alpha_n a^n \Sigma J_n(ka) \Bigg\{ D_k \Big(\frac{h \sinh kh}{k} - \frac{1}{k^2}\cosh kh + \frac{1}{k^2} \Big) + E_k \Big(\frac{h \cosh kh}{k} - \frac{1}{k^2}\sinh kh \Big) \Bigg\} - \sum_k \Bigg\{ \alpha_n D_k \frac{na^n}{k^2} J_n(ka) + \tfrac{1}{2}ka^2 E_k D_k \Big(1 - \frac{n^2}{k^2a^2} \Big) [J_n(ka)]^2 \Bigg\} \Bigg]. \qquad (37)$$

The constants E_k and D_k are very small compared to α_n. If we neglect E_k and D_k, the expression for the kinetic energy reduces to the simple form

$$T = \frac{\pi}{2} \rho n \alpha_n^2 a^{2n} \frac{h^3}{3} \cos^2 pt. \qquad (38)$$

In this case the expression for Φ reduces to the form

$$\Phi = \alpha_n z r^n \sin n\theta \cos pt. \qquad (39)$$

This expression represents the principal mode of vibration of the liquid and all the other coexistent modes are very small compared to this one. Since the expression for the velocity varies as $(r/a)^{n-1}$, the velocity is very marked near the margin of the vessel and is almost imperceptible near the center. Using the principle that the sum of the kinetic and potential energies of the solid and liquid together must be independent of the time, we easily obtain an expression for the frequency of vibrations in the most general case from the expressions for the kinetic and potential energies already given. If we neglect E_k and D_k, the frequency equation takes a very simple form. The expression in this case is

$$[\sigma\tau la\{\tfrac{1}{3}l^2(1 + n^2) + n^{-2}a^2\} + \tfrac{1}{3}\rho n h^3 a^2]p^2 = \frac{8\mu\tau^3 l}{3a}(n^2 - 1)^2 \Big[\frac{\lambda + \mu}{\lambda + 2\mu} \frac{n^2 l^2}{3a^2} + 1 \Big]. \qquad (40)$$

Thus we see that the law of variation of the frequency with the height of water in the vessel can be expressed in the form

$$p^2 = \frac{1}{A + B\,(h/l)^3},$$

where A and B are two constants for the vessel.

For a glass cylinder whose dimensions are given by $l/a = 4$, $\tau/a = .02$ and which has the density $\sigma = 2.6$ and the elastic constants $\mu = 1.8$ and $\lambda = 1.53$, we easily find that the frequency p_n is given by

$$\left[.052\left\{(1+n^2) + \frac{.1875}{n^2}\right\} + n\left(\frac{h}{l}\right)^3\right] p_n^2 = \frac{8\mu\tau^3}{3a^2l^2}(n^2-1)^2\,[10.4\,n^2+3].$$

For the three gravest tones given by $n = 2$, $n = 3$ and $n = 4$, the values of the frequencies p_2, p_3 and p_4 with different quantities of water in the cylinder are shown in Table II.

TABLE II.

h/l.	$p_2 \times$ Const.	$p_3 \times$ Const.	$p_4 \times$ Const.
0	12.47	34.44	65.63
.1	12.33	34.40	65.48
.2	12.02	33.98	64.47
.3	11.29	32.05	61.95
.4	10.15	29.45	57.79
.5	8.85	26.26	52.45
.6	7.61	22.99	46.68
.7	6.56	19.97	41.09
.8	5.60	17.34	36.05
.9	4.97	15.13	31.67
1.0	4.21	13.25	27.93

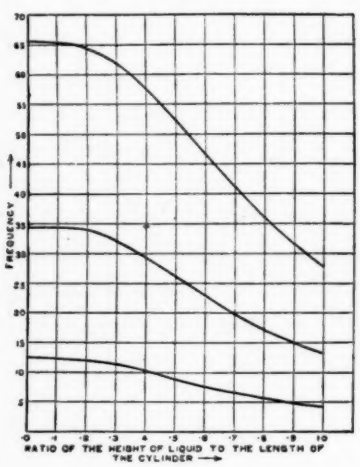


Fig. 2.

The values of the frequencies given in Table II. have been plotted in Fig. 2.

4. CONICAL CUPS.

It is shown in Lord Rayleigh's Theory of Sound in the article already referred to that if a cone for which $\rho = \tan\gamma \cdot z$, γ being the semi-vertical angle, executes flexural vibrations, the displacements $\delta\rho$, $\delta\phi$, δz at any point whose cylindrical co-ördinates are (ρ, ϕ, z) are given by

$$\delta\rho = n \tan\gamma\,(A_n z + B_n)\sin n\phi,$$

$$\delta\phi = (A_n + B_n z^{-1})\cos n\phi,$$

$$\delta z = \tan^2\gamma[n^{-1}B_n - n\,(A_n z + B_n)]\sin n\phi.$$

If the cone be complete up to the vertex at $z = 0$, then $B_n = 0$, so that

$$\delta\rho = n \tan\gamma \cdot A_n z \sin n\phi,$$
$$\delta\phi = A_n \cos n\phi,$$
$$\delta z = -nA_n \tan^2\gamma \cdot z \sin n\phi.$$

If the displacements in polar coördinates (r, θ, ϕ) be denoted by $\delta r, \delta\theta, \delta\phi$, we easily obtain

$$\delta\phi = A_n \cos n\phi$$
$$\delta r = \delta\rho \sin\gamma + \delta z \cos\gamma = 0,$$
$$r\delta\theta = \delta\rho \cos\gamma - \delta z \sin\gamma = nA_n \tan\gamma \cdot r \sin n\phi.$$

It is easy to see that the potential energy of deformation for a length l of the cone

$$W = \frac{4\pi}{3}\mu\tau^3 \frac{\lambda+\mu}{\lambda+2\mu} A_n^2 \sin\gamma \left[\left(-n^3\frac{\tan\gamma}{\sin^2\gamma} + n\tan\gamma + n\cot\gamma\right)^2 + \cos^2\gamma\right] \log\frac{2l}{\tau},$$

where τ = thickness.[1]

The expression for the kinetic energy of vibration of the shell can be easily obtained in the form

$$T = \frac{\pi}{8}\sigma\tau l^4 \sin^3\gamma \,[n^2\sec^2\gamma + 1]\left(\frac{dA_n}{dt}\right)^2. \tag{45}$$

If the cup contains frictionless incompressible fluid, the velocity potential of the fluid must satisfy Laplace's equation. Let us assume that the velocity potential is given by

$$\Phi = C_n r^2 \phi_n (\cos\theta) \sin n\phi \cdot \cos pt \tag{46}$$

where ϕn $(\cos\theta)$ is a function of θ only. It is easy to see by substitution in the differential equation

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta} + \frac{\operatorname{cosec}^2\theta}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0,$$

that ϕn $(\cos\theta)$ satisfies the equation

$$\frac{\partial^2\phi_n}{\partial\theta^2} + \cot\theta\frac{\partial\phi_n}{\partial\theta} + (6 - n^2\operatorname{cosec}^2\theta)\phi_n = 0.$$

[1] This expression can be readily deduced from a very general expression for the potential energy due to strain in curvilinear coördinates obtained by Prof. Love. (Vide his paper on "The Small Free Vibrations and Deformation of a Thin Elastic Shell," Phil. Trans., Vol. 179, 1888, A.) The expression has been criticized by Prof. Basset (Phil. Trans., Vol. 181, 1890, A) on the ground that Prof. Love has omitted several terms which involve the extension of the middle surface. As the inextensional vibrations only have been considered in this paper, this criticism does not affect us in any way.

A solution of this differential equation can be easily obtained in the form

$$\phi n\,(\cos\theta) = \tan^n \tfrac{1}{2}\theta\left[\,(1-n)(2-n) - 6(2-n)\cos^2\frac{\theta}{2} + 12\cos^4\frac{\theta}{2}\right].$$

The relation between C_n and A_n can be easily obtained by equating the value of $\partial\Phi/r\partial\theta$, when $\theta = \gamma$, to $d(r\delta\theta)/dt$, both of which represent the normal velocity at the boundary. We thus get

$$C_n \cos pt \frac{\partial}{\partial\gamma}\left[\tan^n\tfrac{1}{2}\gamma\left\{(1-n)(2-n) - 6(2-n)\cos^2\frac{\gamma}{2} + 12\cos^4\frac{\gamma}{2}\right\}\right] = n\tan\gamma\frac{dA_n}{dt}. \quad (48)$$

The principal mode of vibration of the liquid will therefore be expressed by (46) except for a small correction to be introduced on account of the existence of a free surface. At the free surface the condition to be satisfied is given by

$$\Phi = 0, \text{ when } z = h,$$

where h denotes the height of the liquid.

To satisfy this condition, we assume

$$\Phi = C_n r^2 \phi_n\,(\cos\theta)\sin n\phi\cos pt + \sum_m D_m r^m P_m{}^n\,(\cos\theta)\sin n\phi\cos pt, \quad (49)$$

where the summation extends for all values of m which are the roots of the equation

$$\frac{\partial}{\partial\gamma}P_m{}^n(\cos\gamma) = 0. \quad (50)$$

The constants D_m's have to be determined by means of the equation

$$C_n(h\sec\theta)^2\phi_n\,(\cos\theta) + \sum_m D_m(h\sec\theta)^m P_m{}^n\,(\cos\theta) = 0, \quad (51)$$

which must be satisfied for all values of θ between the limits $0 < \theta < \gamma$. Approximate values of the constants D_m's can be easily obtained from this equation. To get an idea of the magnitude of the constant D_m, we shall obtain its value in the particular case when the semi-vertical angle γ of the cone is small and the height h of the liquid is large compared to the radius of the cross-section of the cone by the free surface. In this case the free surface can be taken to be practically coincident with the surface of the sphere $r = h$. The equation for determining D_m is then

$$C_n h^2\phi_n\,(\cos\theta) + \sum_m D_m h^m P_m{}^n\,(\cos\theta) = 0.$$

Now since

$$\int_{\cos\gamma}^{1} P_m{}^n\,(\cos\theta)P_m{}^n\,(\cos\theta)\sin\theta d\theta = 0,$$

m, m^1 being two different roots of the equation (50),
and

$$\int_{\cos\gamma}^{1} [P_m^n(\cos\theta)]^2 \sin\theta d\theta = \frac{1-\cos^2\gamma}{2m+1} P_m^n(\cos\gamma) \frac{\partial^2}{\partial m \partial \cos\gamma} P_m^n(\cos\gamma),$$

we easily get

$$D_m = -\frac{2m+1}{1-\cos^2\gamma} \frac{C_n h^{2-m} \int_{\cos\gamma}^{1} \phi_n(\cos\theta) P_m^n(\cos\theta) \sin\theta d\theta}{P_m^n(\cos\gamma) \frac{\partial^2}{\partial m \partial \cos\gamma} P_m^n(\cos\gamma)}.$$

Neglecting the small correction introduced by the free surface, we see that the kinetic energy of the fluid motion is

$$\frac{\pi}{2} \rho C_n^2 \cos^2 pt \phi_n(\cos\gamma) \frac{\partial \phi_n(\cos\gamma)}{\partial\gamma} \sin\gamma \sec^5\gamma \frac{h^5}{5}.$$

Since the sum of the kinetic and potential energies of the solid and liquid together must be independent of the time, we easily obtain, on assuming that $A_n \alpha \cos pt$, the frequency equation in the form

$$\left[\tfrac{1}{4}\sigma\tau l^4 \sin^3\gamma (n^2 \sec^2\gamma + 1) + \rho n^2 \tan^2\gamma \sin\gamma \frac{\phi_n(\cos\gamma)}{\frac{\partial}{\partial\gamma}\phi_n(\cos\gamma)} \frac{H^5}{5} \right] p^2$$

$$= \frac{8}{3}\mu\tau^3 \frac{\lambda+\mu}{\lambda+2\mu} A_n^2 \sin\gamma \left[\left(-n^3 \frac{\tan\gamma}{\sin^2\gamma} + n\tan\gamma + n\cot\gamma \right)^2 + \cos^2\gamma \right] \log \frac{2l}{\tau}.$$

where $H = h \sec\gamma$, H being the slant height of the liquid. In this case, we see that the law of variation of frequency with the height of liquid can be expressed in the form

$$p^2 = \frac{1}{A + B(h/l)^5},$$

A and B being two constants for the particular shell.

The frequencies p_2, p_3 and p_4 with different quantities of water for the three gravest modes of vibrations given by $n = 2$, $n = 3$ and $n = 4$ have been calculated from this expression for a cone of semi-vertical angle 30°, the ratio of the thickness of the sides of the cone to the slant height being equal to .02 and are shown in Table III.

The curves showing the fall of frequency for these three modes of vibrations of the cone when loaded with different quantities are plotted in Fig. 3.

TABLE III.

H/l.	$f_2 \times$ Const.	$f_3 \times$ Const.	$f_4 \times$ Const.
0	5.030	13.58	26.75
.1	5.030	13.58	26.75
.2	5.028	13.57	26.73
.3	5.008	13.54	26.69
.4	4.937	13.41	26.47
.5	4.761	13.08	25.94
.6	4.428	12.43	24.87
.7	3.942	11.64	23.13
.8	3.399	10.07	20.79
.9	2.808	8.63	18.07
1.0	2.310	7.26	15.41

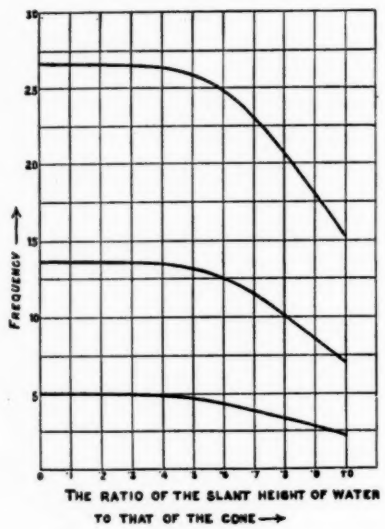


Fig. 3.

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THE RELATION OF POTENTIAL DISTRIBUTION TO HYSTERESIS EFFECT IN THE WEHNELT TUBE.

By R. A. Porter.

1. Introduction.

As a part of his extended study of the discharge in a vacuum tube having a hot oxide as cathode, Wehnelt[1] published a Braun-tube oscillograph curve showing the general character of the voltage and current values in the tube when it is connected to an alternating current generator. Later I made a large number of oscillograph records and from them a study of the relationship of voltage to current in the same form of tube, the cathode being covered with calcium oxide.[2] The curves published at that time show the characteristics for static voltage conditions and for various frequencies of alternating current as gas pressure, maximum current, temperature of cathode, and kind of gas in the tube are varied.

The curves which I obtained showed quite a variety of forms, but one of the marked features was the different current values obtained for a single anode-cathode voltage under both static and alternating current (dynamic) conditions. In general, the static and the dynamic characteristics showed a hysteresis effect, that is, larger current values for decreasing than for increasing voltages.[3] The present paper gives the results of a study of the potential distribution under which three values of current are obtained for the same anode-cathode voltage. The work was undertaken for the purpose of obtaining further knowledge of the conditions producing the hysteresis effect.

2. Apparatus.

The discharge tube used for this study was in form and dimensions almost identical with that of the tube which I used previously in studying the dynamic characteristics, it being essentially the commercial form of Wehnelt tube manufactured by Gundelach, and was obtained from him. It is spherical in shape, 14 cm. in diameter. The only modification from

[1] A. Wehnelt, Ann. d. Phys., 19, p. 138, 1906.

[2] R. A. Porter, Ann. d. Phys., 40, p. 561, 1913.

[3] See figures 4, 15, 10, 11, etc., *l. c.*

the former tube used consists in the addition of a movable sound, shown at *A* in Fig. 1. This sound is made of platinum wire sheathed with glass up to a point within 4 mm. of the end. The other end of the platinum wire is connected electrically to the terminal outside the tube by a coiled strip of bronze. The iron band *L*, attached to the glass tubing, allows the sound to be moved magnetically The anode, *K*, is of iron about 0.22 cm. in diameter and extends about 2.5 cm. within the tube. The end of the anode is 4.28 cm. from the plane of the cathode strip.

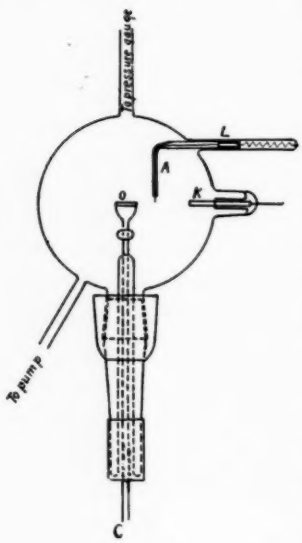

Fig. 1.

The cathode strip, *O*, is platinum, 1.5 × 0.25 × 0.0015 cm. With a heating current or 11.65 amperes, the temperature as obtained with a Wanner pyrometer was 1285° C. with no discharge current, the pressure being 0.010 mm. With a discharge current of 30 milamperes, the temperature read 20° higher. The calcium oxide coat was obtained by placing Kahlbaum c.p. calcium nitrate upon the platinum strip and heating it electrically in the air.

The negative terminal of the storage battery supplying the heating current was grounded at the base of the tube, at *C*. This junction was also the point connected to the electrometer in making measurements of the potential drop from sound to cathode. All potentials were measured finally with the electrometer. A potentiometer method described by Tate and Foote[1] was tried and found to work very well for the larger currents. However, as potentials for the non-luminous discharge had to be measured with an electrometer, it seemed best to use it for all measurements.

A resistance of 1,000 ohms was used in the anode-cathode circuit to stabilize the discharge current, which was furnished by a 300-volt d.c. generator. The applied voltage was measured with a Weston voltmeter, the discharge current, with a Weston millivoltmeter used an an ammeter.

During the observations here recorded, no trap was used to prevent mercury vapor from passing from the pump or the McLeod gauge into the discharge tube.

[1] John T. Tate and Paul D. Foote, Jour. Washington Acad. Sci., 7, p. 482, 1917.

3. POTENTIAL DISTRIBUTION AND STATIC CHARACTERISTICS.

After a number of voltage and current measurements had been made with this tube, three characteristic and readily reproducible stages in the discharge were recognizable, the non-luminous discharge and two forms of luminous discharge. Some transient effects were observed and further study may disclose the conditions under which these are obtained.

The wide difference in the characteristics of such a tube for luminous and for non-luminous discharge have already been pointed out.[1] Child[2] has made potential measurements for the luminous and for the non-luminous discharge in a cylindrical tube with CaO cathode. Lester[3] found that with a CaO cathode "there were generally three possible ranges for the current-potential curves" and that "the current broke suddenly from one range to another." A knowledge of the distribution of potential in the space between anode and cathode for the three forms of discharge observed in the Wehnelt type of tube seemed important for the explanation of the phenomenon of hysteresis as well as that of a multiple valued current for a single anode-cathode potential.

The measurements thus far made have been taken at a pressure of

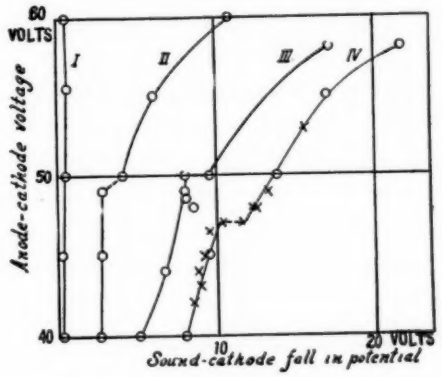

Fig. 2.

Distance of sound from cathode for curve *II*, 0.60 cm., for curve *III*, 2.40 cm.; for curve *IV*, 3.96 cm. These curves are obtained with luminous discharge. Curve *I* shows approximately the sound-cathode fall for non-luminous discharge for all three positions of the sound. Due to change in position, the sound voltage for non-luminous discharge does not vary more than 0.7 volt.

0.010 mm. With this pressure and a cathode temperature in the neighborhood of 1290° C. (heating current 11.65 amperes), either a luminous or a non-luminous discharge could be obtained for all anode-cathode

[1] R. A. Porter, Ann. d. Phys., 40, p. 583, 1913; Richardson and Bazzoni, Phil. Mag., 32, 426, 1916.

[2] C. D. Child, PHYS. REV., 32, 492, 1911.

[3] H. Lester, Phil. Mag., 31, 549, 1916.

voltages between 40 and 60 volts. Whether the discharge is luminous or non-luminous depends on whether this region of voltage is approached from a voltage above 60 or from one below 40 volts. Curves showing the relationship between anode-cathode voltage and sound-cathode fall in potential for three positions of the sound are given in Fig. 2. The sound lay at a point on a line connecting the anode with the middle of the cathode.

The curves of Fig. 3 show more readily the relation of the sound-cathode fall to the distance from the cathode for non-luminous and for luminous discharge for the anode-cathode voltages indicated by the figures at the right of the curves. The three curves of group I. were obtained with the non-luminous discharge; curves II., III., IV. and V. with luminous. A similar set of curves has been made for a heating current of 10.65 amperes. The only striking difference is that for the smaller heating current the potentials near the anode are more noticeably lower than that midway between anode and cathode, a condition which Thomson[1] has observed in a cylindrical form of tube. The apparent irregularity in the readings in the neighborhood of 48 to 50 volts is more pronounced at the higher temperature of the cathode.

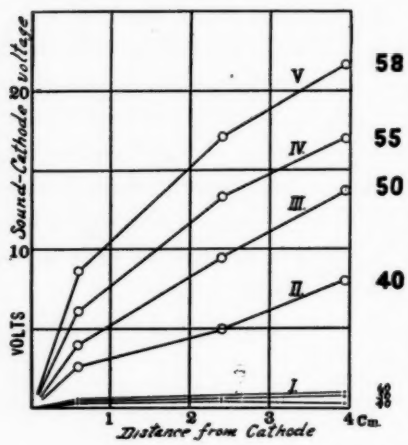

Fig. 3.

Potential distribution for non-luminous and for luminous discharge in Wehnelt tube.

Fig. 4 shows the relation between anode-cathode voltage and current for luminous discharge.[2] The relatively large change in current at an applied voltage of about 50 volts suggests a change in the mechanism of the discharge at this point, just as the wide difference in the values of current for the non-luminous and the luminous discharge for the voltage range 40 to 60 volts indicated a difference in the process of conduction for those two forms of discharge. Observation of the appearance of the glow in the tube also indicates a transition as the voltage goes through the region 48–50 volts. When the glow is present with smaller voltages, it is a faint glow extending from anode to cathode, not filling the whole

[1] J. J. Thomson, Phil. Mag., 18, 441, 1909.

[2] In the articles by Richardson and Bazzoni and by myself, Phil. Mag., 32, 428, 1916, and Ann. d. Phys., 40, 583, 1913, respectively, voltage-current curves for tubes with CaO cathodes are given for both the luminous and the non-luminous discharge.

space inside the tube. It is most intense in the space directly between anode and cathode, where one might expect the current density due to the pure electron (non-luminous) discharge to be greatest, and decreases in intensity and disappears in the region where the current density may be less.

At a voltage of 48–50 volts, however, if the applied voltage is being

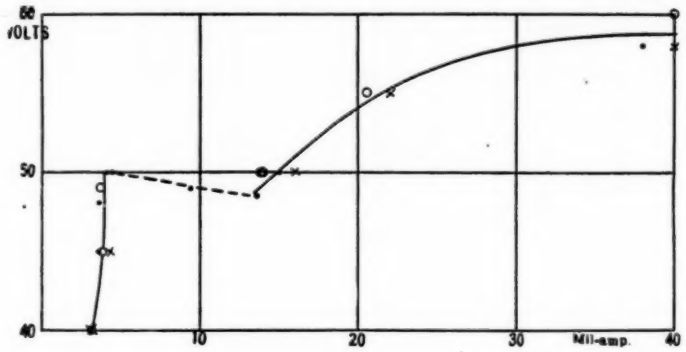

Fig. 4.

Static characteristic for luminous discharge.

increased, a flaming bluish glow appears on the sides of the anode and simultaneously the anode-cathode voltage may decrease slightly and the current increase. As the voltage is raised, the anode light increases in extent and the general glow extends until it fills the whole space inside the walls of the tube; then the anode light contracts and at a higher voltage disappears leaving the whole space inside the tube almost uniformly luminous. After the anode light appears, the current increases very rapidly for small increases in applied voltage. In fact the anode-cathode voltage seemed to remain constant for a considerable range in current. When 50 volts is reached voltage and current rise together.

As the anode-cathode voltage is decreased from 60 volts with a luminous discharge, it may be brought below 50 volts before the anode light disappears. When about 49 volts is reached the discharge changes to the form of a general glow, the anode-cathode voltage rises perhaps a volt and at the same time there is a marked decrease in current.

For one value of pressure and heating current a set of measurements has been made of the anode-cathode voltage, the current, and the potential fall in the line between anode and cathode, for the two stages in the luminous discharge. The curves of Fig. 5 show the relationship of sound-cathode fall to the total anode-cathode fall for three positions of the sound for voltages near 50 volts. Points marked with circles were obtained when the anode-light stage had been reached, those marked by dots show the sound-cathode fall when there is a general glow.

The points on curve I. at 49 volts indicate a considerable variation in the sound-cathode fall for that voltage when the anode light is present. Readings in a single set of observations, however, show regularly that

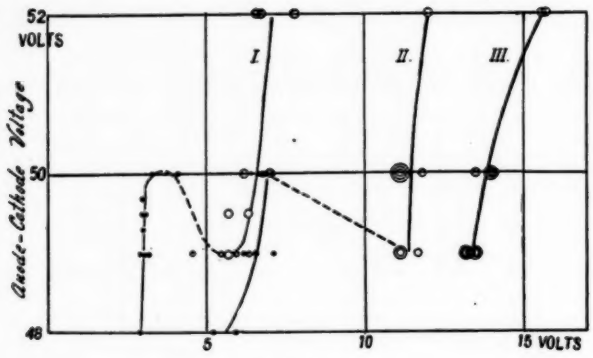

Fig. 5.

Sound—Cathode Voltage. For curves *I* and *II*, *III* the respective distances of sound from cathode were 0.62, 1.96 and 3.93 cm.

there is a difference of two to three volts in this fall for the two forms of discharge with an applied voltage of 49.

The smaller sound-cathode fall occurs when there is a general glow only partly filling the tube. This is the form of luminous discharge occurring first as the voltage is increased and may be called stage II. of the discharge, stage I. being non-luminous. When the region of 50 volts

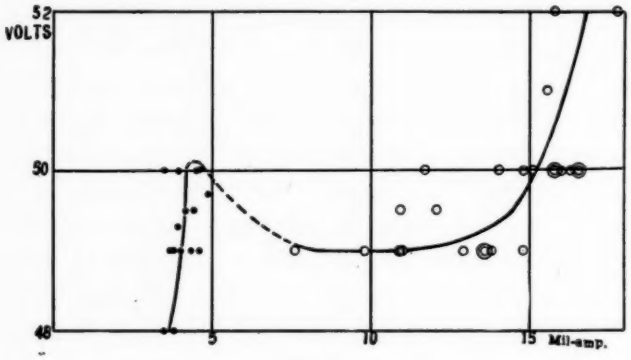

Fig. 6.

Static characteristic of Wehnelt tube in region of 48–52 volts.

is approached from higher voltages, the blue anode light is present and is accompanied by a larger sound-cathode fall and larger currents. We may call this stage III.

The relation of current to voltage for the region 49 to 50 volts is shown in Fig. 6. The potentials existing in the tube for the three stages in the discharge are brought together in Fig. 7.

Readings at higher cathode temperatures indicate an even more sharply marked region of voltage for which the three forms of discharge already noted may occur. At higher temperatures, stages II. and III. apparently have a wider range of voltage in common and there is a larger change in current as the discharge changes from one stage to the other. Modifications of the tube now being used will be necessary in order to obtain a full set of readings at higher cathode temperatures.

When the sound was near the anode, the anode light appeared and disappeared more gradually; no sharp limit of the two forms of discharge

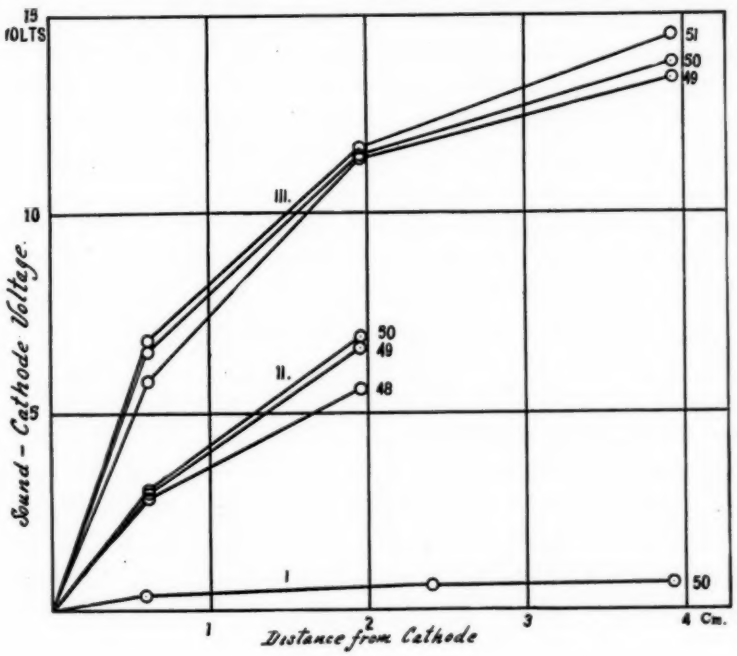

Fig. 7.

Potential distribution for three stages in discharge of Wehnelt tube. The figures at the right of the curves give the anode potentials.

could be distinguished. The curves for this stage in the discharge showing the relation between potential and distance from the cathode in Fig. 7 have therefore been drawn for only about half the anode-cathode distance.

From the results above given it can be seen that the following changes accompany the transition of the discharge from stage I. to stage II. and from stage II. to stage III.: (1) An increase in the cathode fall of 2 to 3 volts; (2) a small increase in the approximately uniform field in the free space between anode and cathode; (3) a decrease in the anode fall; (4) a decrease in the anode-cathode voltage; and (5) an increase in the discharge current. Each of these changes is consistent with the assumption

that there is an increase in the supply of positive ions in the region of the anode, that is, a more vigorous ionization at that terminal, as the current increases.

4. THE HYSTERESIS EFFECT.

The direct observation of the discharge and the study of the dynamic characteristics and of the oscillograph records previously obtained show that during each half-cycle of the alternating current which has a suitable maximum value and takes place under the same conditions of pressure and cathode temperature as here used, all the stages of the discharge here described exist. We may conclude therefore that the widely different values of current at the beginning and at the end of a half-cycle, *i. e.*, just before the beginning and just before the end of the luminous discharge, correspond with entirely different potential distributions of the anode-cathode voltage and hence to different localizations of charges and of ionization.

When the anode light, or stage III., appears in the cycle, there is another abrupt change in the ionization and in the distribution of charge. At each of these transition stages, the old conditions tend to persist. Time and voltage seem to be required to change them.

SUMMARY.

Three stages in the discharge of a vacuum tube with CaO cathode have been identified and potential distribution curves for one pressure and heating current have been obtained for each stage. Voltage-current curves for the two stages of luminous discharge are given. The potential distribution curves lie in three groups corresponding to the three stages of the discharge. With the pressure and cathode temperature used, non-luminous discharge may exist for the voltage range 40–50 volts; another stage in the discharge may begin at 48 volts. In the region of 49 volts, therefore, any one of the three stages of the discharge may be obtained. Each of the three stages of the discharge and potential distribution tends to persist as the voltage is varied, or a hysteresis in the current values and in the potential distribution exists when the potential is cyclically varied. This is apparent for very slow cycles as well as for cycles of higher frequency.

The American Association for the Advancement of Science has granted assistance from its research funds in obtaining apparatus for this work. The author gratefully acknowledges this aid.

SYRACUSE UNIVERSITY,
SYRACUSE, N. Y.,
November 12, 1918.

ELECTRIC DISCHARGE ON THE SURFACE OF A SOLID ELECTROLYTE.[1]

By Was. Šulejkin.

The Theory of Electrolytic Safeties[2] Against Overtension.

As is well known, with the electrolysis of some solutions, the aluminum anode is covered with a coating of oxides in a very short space of time after closing the circuit. At the same time the strength of the current falls almost to zero, as the thin layer of oxides and the layer of gas enclosed in the pores offer an enormous amount of ohmic resistance. It is well known also that by increasing the tension, sometimes by hundreds of volts, the strength of the current is increased only inconsiderably, but this phenomenon remains so only until the tension has passed a certain limit, the so-called "critical tension."

On further increasing the tension, even if to only a small extent, the strength of the current suddenly and sharply rises. On the basis of this has been established the practice of applying the so-called "safety appliances against over-tension," the elements of which (practically) do not allow the current to escape to the ground from the line of transmission of electrical energy when the tension in the line is normal, but immediately and rapidly conducts discharges of electromagnetic waves to the ground, before over-tension in the protected part of the line can arise.

It is customary to say that such "safeties" cannot bear the "strain" when the critical tension is passed; the layer is broken through, and so a direct communication is set up between the fluid and the metallic anode.

But one glance at the aluminum electrodes (placed in a glass vessel to facilitate observation) suffices to show that it is not so. At a tension even considerably below the "critical," bright spots appear on the surface of the electrode (anode), here and there little bubbles of gas are given off; this all shows that here we have a case, not of static, but of kinetic equilibrium. In fact, even at low tension the layer is broken through here and there, but the breaks quickly close up; at a tension above the critical, however, such "cicatrization," for some reason, does not take place.

In order to find out this reason, I tried to use for my experiments the

[1] Reported at the Moscow Institute of Science, May 19, 1918.

[2] Lightning arresters.

same "model" of aluminum electrode as I used in my investigations of the *electrolytic rectifier*[1]. Instead of an aluminum anode, a system of parallelly connected wires of 0.1 mm. was placed into the electrolyte, sealed into glass tubes (Fig. 1*a*). The part of the breaks in which the fluid comes into contact with the aluminum anode was therefore played here by the extremities of the wires placed in the fluid and projecting out of the glass about 0.5 to 0.75 mm.

It proved that on passing the critical tension each tube immediately

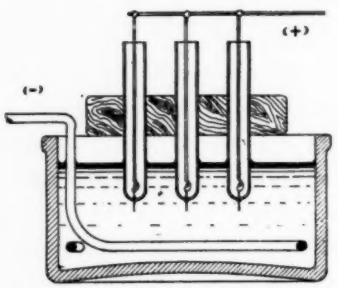

Fig. 1*a*.

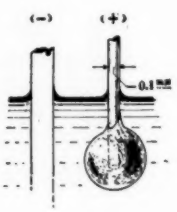

Fig. 1*b*.

broke. On changing the concentration of the solution, as is known, the critical tension point for the safeties is also changed. It was found that this law is followed when the concentration is changed, and that the attainment of the critical tension coincided with the breaking of the glass tubes.

The moment preceding the breaking of the glass, at the bottom end of the tube at the spot where the platinum is fused in, there is a sudden flash of blindingly bright, white light, quite different from the yellowish, weak light of the discharge between the platinum and the solution. When the glass was removed, and an anode, consisting only of fine platinum wires, was put into the solution, no change at all in the character and appearance of the discharge was observed when passing the former "critical " tension; the strength of the current did not sharply increase.

This refutes the opinion of Günt. Schultze,[2] who supposes that at the critical tension the discharge between the metal and the fluid begins to proceed with the help of quite different ions than up to the critical point (analogous to the Glaser-effect[3]). A critical point for discharge between pure metal and the solution, therefore, does not exist.

But, on passing a certain tension, you have only to touch the end

[1] W. Šulejkin, Archives des Sciences Physiques, I., 1918.

[2] Ann. d. Phys., 34, p. 657, 1911.

[3] H. Nernst, Ber. d. Deutsch. Chem. Ges., p. 1547, 1897.

of the wire-anode *under water* with some piece of glass, marble, jasper, enamel, or the like to produce at the point of contact a blinding white light. Such a flash of light appears, as may easily be observed, also at various points of the aluminum electrode (anode), when passing the critical tension.

If the end of the wire is removed to some distance from the glass (or other substance), in the interval between them an electric arc is formed, the anode of which is the wire, while the cathode is the glass (or other substance), to which, as may then be observed, particles of metal, coming away from the anode in a powdered state, are fused.

From this it is clear that the layer, hitherto an insulator, at critical tension begins to conduct the current. A necessary condition for a discharge of this sort is the near proximity of the metal and the conductor of the second kind, both being immersed in the solution and moistened by it. In aluminum safeties this condition is fulfilled; therefore, whenever at any spot the insulating layer is broken through *at a tension above the critical*, there is immediately a characteristic discharging flash at the adjacent part of the layer. At the same time the insulating layer around it is quite destroyed, in just the same way as the glass of the tube, or, for example, the lamina of the jasper. If on any part exposed in this way a fresh layer of oxides should be formed again by the action of the electrolysis, it will immediately be destroyed with a similar discharge. Meanwhile, by each initial rupture the exposure of the metal extends more and more, by which a sharp increase in the strength of the current in the electrolytic safeties is produced, when passing the critical tension.

But a complete resemblance of the work of the aluminium electrode and its "model" may be determined only by observing them under various conditions.

In order, on the one hand, to demonstrate that the critical point characteristic with electrolytic safeties depends on the peculiarities of the layer of oxides, separate parts of which, as it proves, playing a very essential part in the discharge; on the other hand, to become better acquainted with the processes which go on in the layer itself or on its surface; and, finally, to judge in some way of the movement of the ions in the close proximity of the anode, I would pass on to a description of detailed experiments.

I. CRITICAL TENSION WITH SOLUTIONS OF BI-CARBONATE OF SODA OF DIFFERENT CONCENTRATIONS, INTO WHICH AS AN ANODE IS IMMERSED AT FIRST AN ALUMINUM LAMINA AND THEN A WIRE, FUSED INTO GLASS (FIG. 1, *a*).

If we take into consideration that aluminum itself is corroded by the solution (this does not occur in the case of the Pt. wire), which introduces a certain error, it may be seen from the series of figures (see Table I.) that the character of the action of the aluminium electrode and the

TABLE I.

Concentration.	0.6.	0.3.	0.15.	0.075.	0.0375.
Critic. t. of the safety	180 volt	250	280	350	420
Cr. tens. of Pt. wire	164	220	270	340	430

action at the point of contact of our experimental wire with the glass are the same. This gives us a means for determining the critical point even for solutions in which the aluminum anode is not covered with an insoluble layer of oxides (true in the majority of cases); gives us, finally, the possibility of always reproducing the phenomenon in its pure state. The advantage of this will become especially clear if we observe that, for example, with an electrolysis of sulphuric acid between aluminum electrodes the destruction of the insulating layer, *due to the corrosion of the aluminum*, takes place at a tension[1] 100 volts below the real critical tension.

In view of this, further experiments designed to explain the nature of the phenomenon were carried out with "models" of the aluminum electrode.

II. THE DEPENDENCE OF THE VALUE OF CRITICAL TENSION UPON THE CONCENTRATION OF THE ELECTROLYTE FOR DIFFERENT SOLUTIONS.

For experiments, solutions of the following substances have been used: (*a*) Caustic soda (NaOH); (*b*) sodium bicarbonate ($NaHCO_3$); (*c*) iron sulphate ($FeSO_4$); (*d*) zinc sulphate ($ZnSO_4$); (*e*) sulphuric acid (H_2SO_4); (*f*) sodium chloride (NaCl); (*g*) ammonium chloride (NH_4Cl); (*h*) hydrochloric acid (HCl).

Into the electrolyte as an anode were immersed fine wires (platinum or nickel) at the lower ends of which were fused drops of glass of 1 mm. diameter (Fig. 1, *b*).

At first a certain amount of soluble substance was measured out and after each observation the solution was diluted with a double quantity

[1] G. Schultze has called this "seemable critical" tension.

of water, again a double quantity, and so on. In this way a full concentration in gram-equivalents (m) to the liter was determined.

With each concentration the tension was observed at which the bright light flashed out round the part of the glass adjacent to the wire.

The values of P_{crit} were marked off in Descartes coördinates on the axis ordinates; the quantities m were marked off on the axis of abscissa. As by decreasing the concentration by half, again by half, and so on, the critical tension rises, at first slowly, afterwards very rapidly, for greater precision the points were transferred to logarithmic paper (on the abscissa axis was marked off Lgm, on the ordinate LgP_{cr}).

On doing this, a remarkable law became evident: all the characteristics of all the solutions experimented with, without exception, were represented on the logarithmic paper by straight lines.

Let us examine their peculiarities. (a) First of all it will be noticed from the drawing (Fig. 2) that all the straight lines are similar in appearance, being represented by the equation.

$$A \cdot LgP_{cr} + B \cdot Lgm = C, \tag{1}$$

where A, B and C are positive coefficients.
Hence:

$$LgP_{cr} = \frac{C}{A} - \frac{B}{A} Lgm = Lg\alpha - \beta \cdot Lgm \tag{2}$$

(For convenience we have introduced the symbols:

$$\frac{C}{A} = Lg\alpha \text{ and } \frac{B}{A} = \beta).$$

Therefore, it is not difficult to see, for all the solutions:

$$P_{cr} = \frac{\alpha}{m^{\beta}}. \tag{3}$$

(b) As the characteristics of $ZnSO_4$ and $FeSO_4$, of NaCl and NH_4Cl respectively coincide, the height of the critical tension, therefore, does not depend on composition of the cation, but depends only on the anion (SO_4^{--}, Cl^-, OH^-, etc.).

(c) In the diagram it is seen that the critical tensions of the following characteristics also coincide:

NaCl (or NH_4Cl) — to HCl; and $FeSO_4$ (or $ZnSO_4$) — to H_2SO_4.

But it is known that the dissociated acids (HCl, H_2SO_4) are incomparably stronger than their salts. With equal full quantities of the dissolved electrolyte ($m_{acid} = m_{salts}$) in the solution of acid there are incomparably more *dissociated* ions than in the salt solutions. Notwithstanding this, as we see, the critical tension in both cases is the same.

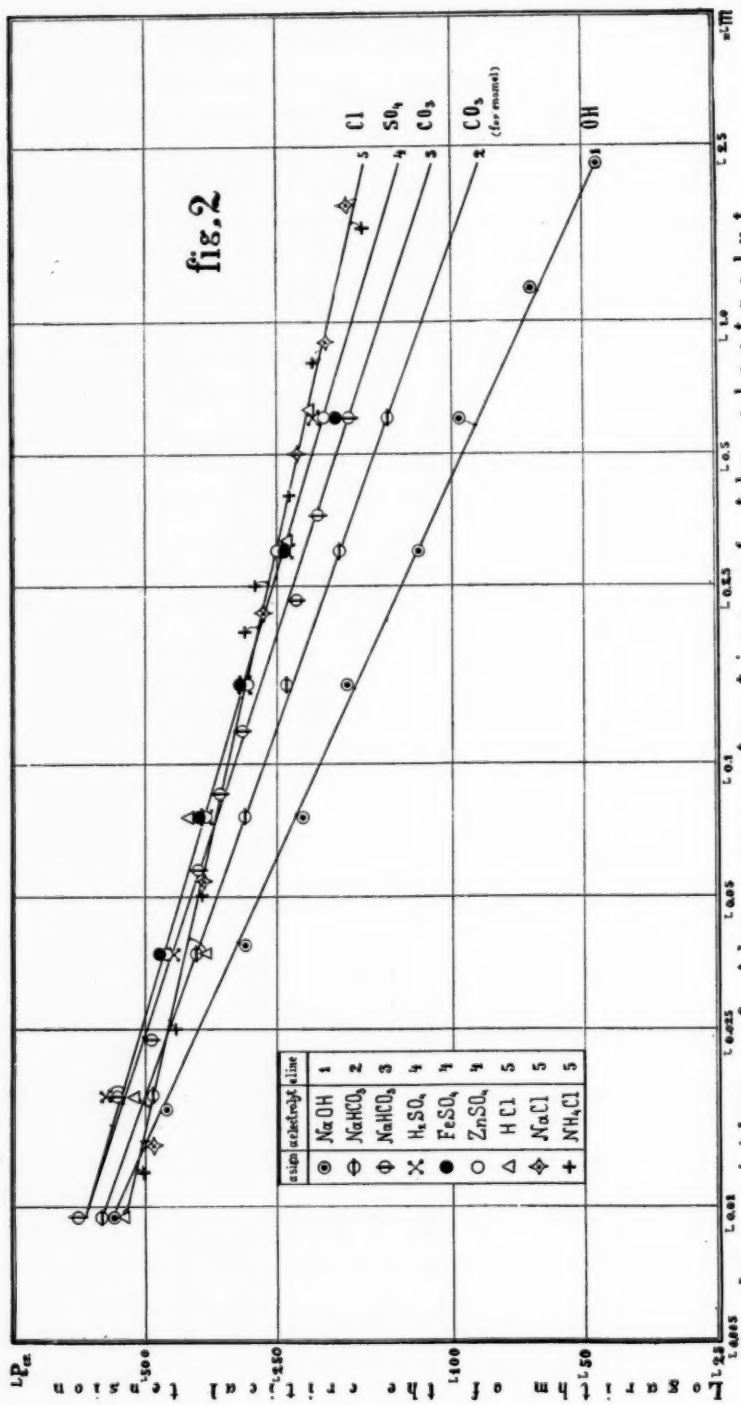
fig.2
Cl
SO4
CO3
CO3 (for mixed)
OH
NaOH 1
NaHCO3 2
NaHCO3 3
H2SO4 4
FeSO4 4
ZnSO4 4
HCl 5
NaCl 5
NH4Cl 5
Logarithme of the concentration of the electrolyt
Logarithm of the critical tension

Fig. 2.

Hence it may be concluded that the value of the critical tension does *not depend* on the number of free ions, but depends on the full amount of the dissolved electrolyte. Evidently, the processes at the anode are extremely violent; at its surface new masses of the electrolyte are continually and rapidly dissociated, depending upon the manner in which the discharge takes place.

Upon the number of free ions depends, it is true, the decrease of tension in the solution, but as the critical tension attains hundreds of volts, the decrease of tension in the solution may evidently be neglected, as compared with the decrease of tension in the close proximity of the anode.

(*d*) As regards the mutual relations of the different characteristics (for the different anions), the following peculiarities are clearly noticeable:

The stronger the acid character of the anion, the less is the inclination of the corresponding straight line on the logarithmic paper (the less is β) and, on the other hand, the higher it intersects the ordinate "Log $m = 0$" ($m = 1$) (the greater is α).

So, from the observations, the greatest α (the smallest β) corresponds to the anion $(\bar{Cl})$, the smallest α (the greatest β) to the anion $(\bar{OH})$.

TABLE II.

No. of a Line.	Anion.	α.	β.
1	$(OH)^-$	68	0.460
3	$(HCO_3)^-$	141	0.336
4	$(SO_4)^{--}$	165	0.304
5	$(Cl)^-$	186	0.230

III. THE DEPENDENCE OF THE HEIGHT OF CRITICAL TENSION ON THE COMPOSITION OF THE SUBSTANCE IN CONTACT WITH THE WIRE.

To the ends of the wires were fused drops of different sorts of glass, of enamel, of jasper, etc. The following results were obtained:

(*a*) The critical tension does not depend on the degree of fusibility or infusibility of the glass in contact with the wire, which was tested by experiments with *six* sorts of glass, from the most fusible to the most insusceptible to fusing.[1]

(*b*) A slight coloring of the glass (transparent glass) does not change the critical tension appreciably. A considerable admixture of metallic oxides (enamel), however, affects the character of the discharge: α decreases considerably, β slightly increases.

For an example the characteristic has been set down on the diagram

[1] Not to obscure the drawing, the points obtained from these experiments are not marked on the straight line.

(N2), corresponding to the little drops of enamel (dark blue or sky blue) fused to the ends of the fine wires immersed in the solution of bicarbonate of soda ($NaHCO_3$). For this case we obtained $\alpha = 114$; $\beta = 0.34$; while for glass drops, $\alpha = 141$; $\beta = 0.336$ (N3). Similar results were obtained from the experiments with drops of jasper.

(*c*) When the wires come into contact with such a crystal as does not possess electrolytic, but metallic[1] conducting properties, our characteristic discharge is not observed.

IV. MICROPHOTOGRAPHS.

In order to be able to determine in some measure the form of the discharge in question, which is not seen by the naked eye, microphotographs were taken. As the strength of the light of the discharge, not withstanding its comparatively great intensity, is still insufficient to give a highly magnified representation in ordinary microphotographic chambers, it was necessary to take the photograph in a somewhat different way. The object was placed within the range of a small microscope (maximum linear magnifying power—60 times). The weakest object glass was used, giving a full linear magnification of 20 times. After the microscope (with external light) had been focused, the eyeglass was unscrewed, and on the diaphragm beneath it (just at the spot where the real image is formed, was placed (with red light) a small square photographic plate (1.5 cm. × 1.5 cm.). The current was turned on (for an instant), and the discharged flash was recorded on the plate, magnified, for example, 5 times. From the miniature negative thus obtained, the final enlargement was then made with the assistance of a projecting lantern.

The first photograph (Fig. 3) shows a discharge at the point of contact of the fine wire with the drop of glass fused to it and immersed in the electrolyte. The discharge encircles the wire with a hoop of light (the points of light on the surface of the drop are only gleams from the refracted and reflected light illuminating the drop).

From the drop a fine hair of glass attached to the wire was drawn upwards along the wire. On the photograph it may be seen that a discharge took place on it also.

The second (Fig. 4) photograph represents a part of the aluminum electrode at critical tension. As the layer of oxides is very thin, it is very rapidly destroyed by the discharges, which spread in rings from the points of incidental rupture. The representation, therefore, is blurred and dull.

[1] As, for example, rock salt, as shown by A. F. Joffe.

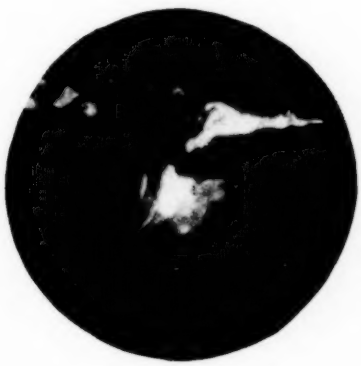

FIG. 3.

FIG. 4.

FIG. 5.

WAS. ŠULEJKIN.

Considerably clearer, however, does the discharge appear round the artificial rupture on the *third* photograph (Fig. 5). The black speck in the middle is a bared portion of the metal. In the adjacent layer of oxides (glass!) there was a flashing discharge, encircling the breach like a ring.

From the material just presented some conclusions regarding the nature of the discharge in question may be drawn. First of all, it is apparent that at the critical tension the path of the discharge changes. Namely, at a lower tension its course is "fluid—film of gas—metal" (A), at a higher tension it is "fluid—oxides—metal" . . . (B). The decrease in tension in each path is so great that henceforth we shall consider it equal to the tension between the electrodes, ignoring the fall in tension in the fluid on the way from the metal cathode to the film near the anode. Combined with such a great fall in tension is the generation of a considerable amount of heat at the surface of the anode, and it is quite natural to assume that the discharge's *change of path* is caused by the heating of the layer of oxides, which reaches a "critical" degree at "critical tension." Point III_c shows that the discharge may take place only at the surface of a solid electrolyte, but not of a crystal possessing metallic conducting properties (rock salt NaCl). Consequently, an essential condition for the formation of a characteristic discharge (change of discharge from path *A* to path *B*) is the increase of conducting power in a body by heating. But this condition alone is not sufficient. In fact, at the same temperature, the mobility of ions, and consequently the conducting power, for example, with easily fusible glass is greater than with less easily fusible glass.

Still, however, we see from III_a that the height of critical tension does not depend on fusibility.

Let us see what indications the external appearance of the discharge may give. The first striking thing of all to observe is that the discharge flashes out suddenly, with a leap, no intermediate stage being noticeable; the portion of glass adjacent to the wire (or the portion of the oxide layer surrounding the point of rupture) all at once becomes heated to white heat. All this reminds us, as nothing could more clearly, of the electric arc phenomenon. This similarity is especially emphasized also by the appearance of the microphotographs *1* and *3* (FIG. 3 and Fig. 5), and the peculiarity III_b. But even by the microphotographs, taken as they are through a certain thickness of fluid, one cannot definitely be convinced that electric arcs really appeared. As these arcs, if they really exist, can arise only in the interval glass—metal (or oxides—metal), I endeavored to make an experiment of the following character:

At the ends of two platinum (or nickel) wires, placed at a distance of 0.5 to 1 cm. from one another, was fixed a "little bridge" of glass tubing, the ends of which were fused to the wires. If you connect the wires[1] to the source of a constant current with a tension of 100–200 volts and place the "little bridge" over the flame of a Bunsen burner, after some time near the wire-anode will appear bright luminous points.

The points continually move about, going out in one place and appearing in another. Through a magnifying glass it may be distinctly seen that these are miniature electric arcs, seldom reaching the length of 1 mm. If we examine the glass "bridge," after such a discharge, it will be seen that the glass around the wire-anode has acquired a spongy construction. Inside the empty spaces which have been formed move the arcs, which, as it were, "lick" the glass.

If the current is not switched off, but only the heating by the gas flame is discontinued, the arcs for some time do not go out, but the smallest movement of the air is sufficient to extinguish them, after which, without outside heating, they do not appear again.

Now, it seems, we may pass on to an explanation of the critical point of the discharge in question.

The phenomenon, most probably, proceeds thus:

Under the influence of the discharges between the fluid and the metal anode, the layer of glass adjacent to the wire is heated, its conducting power is increased, and a part of the current follows the course of fluid^{-}—$^{+}$glass^{-}—$^{+}$metal. But, as shown by Nernst,[2] in the glass near the anode a rapid evolution of gas takes place, which "polarizes" the system, separates the glass from the metal. Therefore, at a tension below "critical," the current within the glass may freely be ignored; the discharge proceeds, practically, by way of the fluid^{-}—$^{+}$metal.

But if the tension is increased more and more, the edges of the glass at last become heated to such an extent that they begin to emit electrons. At this moment the arc appears between the glass and the metal. The tension, accordingly, at which the arc-path, fluid^{-}—$^{+}$glass^{-}—$^{+}$metal, is formed is also the critical tension.

For a sufficient heating of the layer of glass, let there be in its proximity, in the gas separating the fluid from the metal, a generation, under the influence of the discharges, of W joules of heat per second to the unit of surface.

The heat energy of the discharges depends on the fall of tension, P, and on the number of passing ions, q.

[1] Essentially in accordance with a rheostat of about 100 Ω.

[2] Ann. d. Phys.

This latter quantity we cannot determine directly; neither can we judge it by the indications of an ammeter, as its pointer oscillates irregularly; but some observations of the *appearance* of the discharges between the liquid and the metallic wire give reason to *assume* that q is proportionate to P[1] and m.[2]

So we suppose:

$$q = k_1 mP, \tag{4}$$

where k_1 is a coefficient of proportionality.

The fundamental condition W = const. may be set down thus:

$$W = k_2 \cdot P \cdot q = k_1 \cdot k_2 \cdot mP^2 = \text{const} \tag{5}$$

or

$$mP^2 = \text{const}$$

Hence:

$$P_{cr} = \frac{\alpha}{m^{0.5}} \tag{6}$$

where a is constant.

If we remember our characteristics of discharges, we see that, for example, the dependence of P_{crit} on m for the system glass—metal immersed in a solution NaOH is expressed thus. Instead of an exponent of 0.5, we have only $\beta = 0.46$.

Why do all the characteristics, differing from each other in quantity, present nevertheless such remarkable simplicity? Why is there always between P_{crit} and m such simple gradual dependence, notwithstanding the fact that α and β for different substances are different? Our assumption led us to the exponent $\beta = 0.5$; in its turn Fig. 2 and formula (3) perhaps, will give some indication of how *in reality* the ions of the fluid near the surface of a solid electrolyte move.

Conclusion.

1. The behavior of the aluminum anode *up to* critical tension is explained by a *kinetic*, but not a static equilibrium. In the layer of oxides new ruptures are continually being formed, while the old ones close up.

2. At tensions *above* the critical, the equilibrium is disturbed by the fact that the layer of oxides adjacent to each rupture begins to take part in the discharge, which from the path "fluid^{-}—metal^{+}" passes over to "fluid^{-}—oxides^{+}—metal^{-}."

As in the first case (A), on the way from fluid to metal, so also in the second (B), on the way from oxides to metal, the discharge takes place

[1] It is easy to observe the increase in number of the cluster of discharges with the increase in P or m.

[2] With regard to m, see II_c.

in a thin film of gas (oxygen). In the latter case the discharge has the form of an electric arc, heating the edge of the layer of oxides (cathode of the arc) to white heat, and rapidly destroying it.

3. Such a form of discharge may be produced also artificially. To produce it, a metal conductor (anode), touching *below the surface of the fluid* the surface of glass, marble, enamel, or other so-called solid electrolyte, must be immersed in the electrolyte.

4. The critical tension at which this discharge begins, depends on the concentration of the solution, and on the composition of the anion, the dependence between P_{crit} and m for all the solutions experimented with being expressed by the formula:

$$P_{cr} = \frac{\alpha}{m^{\beta}}. \qquad (3)$$

For the dependence of α and β on the character of the anion Table II.

P_{crit} does not depend on the degree of dissociation.

5. In this article it has been possible to explain only roughly the reason for this simple dependence so clearly obtained from the experiments. For the investigation of the actual movement of the ions near the electrode, it may, perhaps, serve as a thread for persons more competent in questions of electro-chemistry.

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MOSCOW, May, 1918.

MAGNETO-STRICTION WITH SPECIAL REFERENCE TO PURE COBALT.

BY HOWARD A. PIDGEON.

PART I. THE WIEDEMANN EFFECT.[1]

THE inter-relations between magnetization and mechanical strain in the ferro-magnetic metals are widely varied in their nature and some of them are extremely complex in character. Their study has formed such a wide field of investigation that no attempt will be made here to more than briefly outline some of the more salient features of the extensive literature on the subject.[2]

THE JOULE EFFECT.

In 1847 Joule[3] found that the length of a soft-iron bar was slightly increased when magnetized, and at about the same time Matteucci[4] discovered that the longitudinal magnetism in an iron rod was increased by the application of a longitudinal pull. It was later discovered by Vallari[5] that this effect is reversed in sufficiently strong magnetic fields, and is commonly known as the Vallari reversal. The reciprocal relations between magnetization and mechanical strain discovered by Joule and Matteucci were shown by the work of later investigators to extend throughout the field of magneto-striction and have been dealt with from theoretical considerations by J. J. Thomson,[6] Kirchhoff,[7] Heydweiller,[8] Gans,[9] Houstoun[10] and others, who have derived mathematical relations

[1] This is the first of a series of articles in preparation under the general title given above.

[2] For a fuller discussion of these effects the reader is referred to the following: Poynting and Thomson, Electricity and Magnetism; Ewing, Magnetic Induction in Iron and Other Metals; Maxwell, Electricity and Magnetism, Vol. II., 3d edition, pp. 90–94; Wiedemann, Electricität, Vol. 3, p. 519. A fairly comprehensive bibliography of the subject up to that date is given at the end of an article by H. G. Dorsey, PHYS. REV., Vol. 30, p. 718, 1910.

[3] Phil. Mag., Vol. 30, pp. 76, 225, 1847.

[4] Comptes Rendus, 1847.

[5] Pogg. Ann., 1868.

[6] Thomson, Applications of Dynamics to Physics and Chemistry.

[7] Wied. Annalen, 1885, Vol. 24, p. 52.

[8] A. Heydweiller, Ann. d. Phys., Vol. 11, p. 602, 1903.

[9] R. Gans, Ann. d. Phys., Vol. 13, p. 634, 1904.

[10] Phil. Mag., Vol. 21, p.78,1911.

for these reciprocal relations, which, however, have been only very roughly verified by data,[1] due no doubt largely at least to magnetic and elastic hysteresis which play a very important role in all magneto-elastic phenomena.

The first really comprehensive view of the Joule effect, as the change in length due to magnetization is called, is due to the extensive work of Shelford Bidwell[2] who worked with specimens of iron, nickel and cobalt, and also studied the change in the effect produced by the application of longitudinal stress. As a result of his work and also that of more recent investigators, especially of Honda and his co-laborers,[3] the general relations between longitudinal magnetization and strain in iron and nickel have been quite definitely established, and with considerably less certainty in cobalt.

The principal features of the work done may be summarized as follows: soft iron elongates when subjected to a magnetic field of weak or moderate strength, retracts in larger fields and in strong fields becomes shorter than its original length. The effect of pull is to increase the intensity of magnetization in weak or moderate fields, and to decrease it in strong fields. The results for steel are similar, but the initial magnetic elongation is less in general.

In the case of nickel, longitudinal magnetization is always accompanied by a decrease in length which approaches an asymptotic value in strong fields, while the effect of longitudinal tension is to diminish the magnetization for all values of the magnetic field.

In the case of cobalt the results have been far less conclusive. Bidwell[4] experimenting with a short rod of cast cobalt found a decrease in length with increasing magnetic field, followed by a retraction reaching the original length at about 750 gauss, and in still stronger fields the specimen became longer than its original length. Thus we see that the effect in cobalt was found to be the exact reverse of that in iron. This result was later confirmed by Nagaoka and Honda[5] for cast cobalt, but a similar annealed specimen was found to decrease in length gradually and did not reach a maximum shortening even in a magnetic field of 2,000 gauss. It was found as expected that the effect of longitudinal tension upon the cast cobalt was to diminish the magnetization in fields of moderate strength and to increase it in very strong fields, while longi-

[1] Honda and Tereda, Phil. Mag., Vol. 14, 1907, pp. 65–107.

[2] Proc. Roy. Soc., 1886, Vol. 40, pp. 109, 257; Phil. Trans., 1888, Vol. 149, p. 205; Proc. Roy. Soc., 1890, Vol. 47, p. 469.

[3] Phil. Mag., 1898, Vol. 46, p. 261; 1900, Vol. 49, p. 329; 1902, S. 6, Vol. 4, pp. 45, 338, 459, 537; 1903, S. 6, Vol. 6, p. 392; 1905, S. 6, Vol. 10, p. 548, 642.

[4] Phil. Trans., vol. 149.

[5] Phil. Mag., Vol. 4, p. 51, 1902.

tudinal pull always diminished the intensity of magnetization in annealed cobalt.

Now all of these specimens were undoubtedly of very impure cobalt, so there has remained some doubt as to what the character of the effect would be in specimens of pure cobalt, since it is known that a marked change may be produced by relatively small quantities of impurities. Bidwell gives no analysis of his specimen but makes the statement that it was soft and easily worked which indicates a considerable amount of impurity since pure cobalt is hard and brittle and not easily worked without special heat treatment. Honda,[1] however, gives an analysis of his specimens which contained from four to five per cent. of nickel, more than one per cent. of iron, and a very high carbon content of from 1.38 to 1.64 per cent.

It was shown by the work of Kelvin,[2] Ewing[3] and others that the effect of longitudinal compression upon magnetization is just the reverse of that produced by longitudinal pull, and that the effect of transverse stress upon longitudinal magnetization is just the reverse of that produced by longitudinal stress.

The work of Nagoaka and Honda[4] has shown that magnetization is also attended by a change in volume which is, in general, of a much smaller order than the change in dimensions.

The Wiedemann Effect.

Some years previous to Joule's discovery Wiedemann[5] experimenting with iron wires made the interesting discovery that when the specimen was suspended in a very small vertical magnetic field, while at the same time an electric current flowed through the specimen, its free end was observed to twist in such a direction that to an observer looking along the specimen in the direction of the flow of current and also in the direction of the magnetic lines of force, the lines of twist were in the direction of a right-handed screw, that is the twist was positive in direction. The direction of twist was reversed upon the reversal of either the current or the longitudinal field. If the current in the specimen remained constant and the longitudinal field was gradually increased, the twist reached a maximum in fields of from 15 to 36 gauss, gradually decreased in stronger fields until in very strong fields the direction of twist changed and became negative or left handed. This is commonly known as the Wiedemann effect.

[1] Phil. Mag., S. 6, Vol. 4, p. 48.
[2] Kelvin, Reprint of Papers, Vol. II., pp. 332–407.
[3] *Loc. cit.*, p. 202.
[4] Phil. Mag., S. 5, Vol. 46, p. 261; S. 5, Vol. 49, p. 329.
[5] Electricität, Bd. 3.

Wiedemann also found that the reciprocal relations already referred to hold here too, for if a wire in which a current was flowing was twisted it was found that longitudinal magnetization was developed, or if a specimen was subjected to a longitudinal field and then twisted circular magnetization resulted.

Investigation by Knott,[1] and Nagoaka and Honda[2] with nickel wires showed that the twist was always negative in direction and much larger than in iron.

The inability to draw cobalt into the form of wires until recently proved an almost insurmountable obstacle to the study of the Wiedemann effect in that metal. However, Honda and Shimizu[3] did succeed in making some measurements on two cobalt rods 21 cm. in length and approximately one centimeter in diameter. They found that for cast cobalt the twist in small magnetic fields was in the same direction as for nickel. It reached a maximum in a somewhat higher field and then decreased, finally reversing its direction in very strong fields. The magnitude of maximum twist was found to be approximately the same as that in similar specimens of iron. In the case of annealed cobalt the twist was very small and did not reach a maximum until a field of about 200 gauss was reached; it then gradually decreased but did not reverse even in very strong fields.

Beside the uncertainty due to the use of impure cobalt, there is still an additional uncertainty here because it was found that the curves obtained for the twist in similar rods of iron and nickel varied considerably from those obtained when wires approximately one millimeter in diameter were used. It would therefore seem especially desirable in this case to obtain data using specimens of as nearly pure cobalt as possible. Although there are many other interesting magneto-elastic effects we shall not discuss them here since the Joule and Wiedemann effects are the fundamental ones and the only ones studied in this work.

As the result of a detailed study of his own work and that of others, Kelvin was able to explain many of the magneto-elastic effects on the basis of magnetic æolotropy produced by stress. In the case of iron in moderate fields he showed that the effect of a simple pulling stress is to produce a greater permeability along than across the lines of strain, while a compressional stress has just the opposite effect. Above the Vallari reversal point the effect is reversed. Kelvin further showed that the idea of magnetic æolotropy may be employed to explain the Wiedemann effect and its reciprocal relations. In this case by the super-position of

[1] Trans. Roy. Soc. Edin., Vol. 32, p. 193, 1883.

[2] Phil. Mag., S. 6, Vol. 4, p. 61.

[3] Phil. Mag., S. 6, Vol. 5, p. 650.

the circular field due to the current in the specimen, upon the longitudinal field, the direction of the resultant field in any given element is a diagonal lying in a plane tangent to the element and making an angle with the transverse plane, which varies from zero at the center to a maximum at the circumference. According to Kelvin's theory this must result (for moderate fields) in an elongation in the direction of the resultant field and a shortening in a direction at right angles to it. Since the change in volume is of a smaller order, there must result a shearing strain in a plane making an angle of 45 degrees with the resultant field and, consequently, this strain will have a component in a direction tangent to the element and in a plane perpendicular to the axis of the wire. Such a strain must result in a twist of the specimen. Applying similar reasoning one can predict the nature of the effect produced by stress upon circular or longitudinal magnetization.

Although the conception of magnetic æolotropy is sufficient to explain many of the magneto-elastic effects qualitatively at least, after making due allowance for hysteresis effects, it does not afford a satisfactory explanation of many of the experimental facts.[1] For example, according to this theory the Wiedemann effect may be regarded as only a special case of the Joule effect, and one would accordingly expect to find a comparatively simple relation between the two. However, this is not the case, as has been pointed out by S. R. Williams.[2]

OBJECT OF THIS WORK.

The object of this work is two-fold: first, to study the Joule and Wiedemann effects in specimens of pure cobalt wire and if possible to establish a definite relation between the two effects; second, to make a comparative study of these effects in specimens of cobalt, iron and nickel wire previously subjected to exactly the same heat treatment. As has already been indicated, previous investigation of the Wiedemann effect in cobalt has been of such a meager character and all work on cobalt with such impure specimens that it seemed highly desirable to repeat the work with pure specimens now available.

Moreover, as has been pointed out by S. R. Williams,[3] there has been a great lack of coördination in the work done in this field. Much of the experimentation has been upon specimens of whose chemical composition or previous history but little or nothing is known, and since both have a very important influence upon the magneto-elastic effects, much of the data taken has a questionable value so far as making a comparative study

[1] See Ewing's Magnetic Induction in Iron and Other Metals, pp. 244 and 246.

[2] PHYS. REV., Vol. 32, p. 295.

[3] PHYS. REV., Vol. 34, p. 258.

is concerned. It is evidently highly desirable that as many as possible of these effects be studied in the same specimens which have previously been subjected to the same heat treatment. It was with this object in view that a considerable portion of this work was undertaken. This paper will deal with the Wiedemann effect in cobalt and for comparison, also in iron and nickel; while a later paper will deal with the Joule effect and a comparative study of these and other magnetic phenomena.

SPECIMENS.

The specimens of cobalt used were obtained from the laboratory of the School of Mining in Queen's University, through the generosity of Eugene Haanel, director of the Mines Branch, under whose direction H. T. Kalmus and others made an extensive study of cobalt[1] and succeeded not only in producing very pure metal on a commercial basis but also by special heat treatment, in drawing it into wires suitable for many tests otherwise almost impossible. As will be seen from the following analysis given by the School of Mining, the specimens were comparatively pure.

Specimen *A*.	Specimens *B* and *C*.
Co = 99.73	Co = 98.71
Fe = 0.14	Fe = 1.15
Ni = 0.00	Ni = 0.00
C = 9.00	C = 0.039
S = 0.019	S = 0.012
Si = 0.02	P = 0.01
	Si = 0.14

TABLE I.

No. of Specimen.	Material.	Condition.	Length in CM.	Diam. in CM.
A	Cobalt	Annealed in hydrogen.....	25.02	0.0884
B	Cobalt	Unannealed.....	24.92	0.0914
C	Cobalt	Annealed in hydrogen.....	25.06	0.0912
E	Nickel	Annealed in hydrogen.....	25.01	0.1003
F	Nickel	Unannealed.....	25.10	0.1002
H	Iron	Annealed in hydrogen.....	25.00	0.1023

The nickel specimens were from wire labelled "pure nickel" and the iron from the core of an old induction coil, so although no chemical analysis has thus far been made they are undoubtedly of fairly pure material. All of the specimens tested were approximately 25 centimeters in length and varied from 0.85 mm. to 1.00 mm. in diameter. The exact dimensions are given in Table I.

[1] See Bulletin entitled, The Physical Properties of the Metal Cobalt, Canadian Department of Mines, Part II., by H. T. Kalmus and C. Harper.

Specimens A, C, E and H were annealed at 800 degrees centigrade for three hours, after which the temperature was very slowly reduced to that of the room, the process occupying several hours. Oxidation was prevented by keeping a stream of hydrogen flowing through the furnace. After this treatment the iron and nickel were extremely soft and flexible but the cobalt was still hard and showed a decidedly crystalline structure. Cobalt specimen B has the same composition as C but was left in the original condition. Nickel specimen F is the same as E except that it was not annealed. Data was also obtained from specimens of iron and nickel annealed by alternating current, but the results are not given as they did not differ materially from those for specimens E and H.

Apparatus.

The arrangement of apparatus is shown diagrammatically in Fig. 1. The solenoid, S, was attached to the plank, P_1, supported firmly between two massive brick pillars about three feet apart. The specimen, W, was

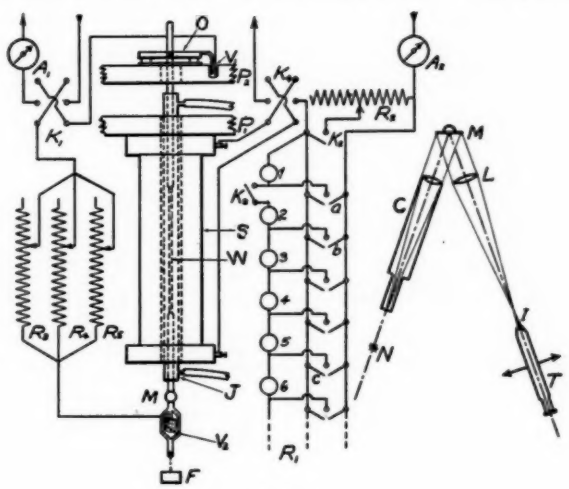

Fig. 1.

soldered to two brass rod extensions and the whole suspended by means of the knife-edge support, O, from the plank, P_2, which was supported by the brick piers entirely independent of the plank, P_1. This was to guard against any possible disturbance due to mechanical displacement produced by the field in the solenoid.

The earth's field was determined and the vertical component compensated for by an additional winding on the outside of the solenoid. As it was found in some cases that the rise in temperature of the specimen due to the heating of the solenoid produced quite an appreciable effect

upon the readings, a double-walled water jacket, J, made of brass tubing was placed inside the solenoid and water from a hydrant kept flowing through it during the experiment. Electrical connection with the specimen was made by means of the mercury cups, V_1 and V_2, into which dipped copper contacts attached to the specimen. By means of a special construction illustrated in the figure, a small weight, F, could be suspended from the specimen without interfering with its freedom of motion. The weight was made just large enough to keep the wire accurately vertical in the central line of the solenoid, and to prevent the weight from vibrating it was suspended in oil at the end of a long flexible cord. The entire suspended system was enclosed to prevent the disturbing influence of currents of air in the room.

As it was desired to measure angles varying from zero to considerably more than one degree in magnitude with an accuracy of at least one second for the smaller angles, a special device was necessary for the purpose. After trying several methods, the very simple one illustrated in Fig. 1 was chosen. Lengthwise in the fairly narrow slit of the collimator, C, was mounted a fine glass fiber. A Nernst glower, N, brilliantly illuminated the slit, rays from which after being rendered parallel by the collimator lens fell upon the plane mirror, M, mounted upon the brass rod attached to the lower end of the specimen. The reflected parallel rays were focussed by an achromatic lens, L, forming an image at I, which was viewed by the traveling microscope, T. This image when viewed through the microscope showed a very well-defined diffraction pattern produced by the double slit, and consisted of a large number of alternate light and dark parallel bands. By special adjustment two or three of these lines could be made to appear predominantly clear and well defined, and upon these very good settings could be made by means of double cross-hairs in the eye-piece of the microscope.

The system was calibrated by replacing the specimen and mirror, M, with another mirror mounted in the axis of a lever approximately 65 cm. in length moved by a micrometer screw. It was found that one division of the micrometer screw of the microscope corresponded to approximately one second of twist of the specimen. By taking an average of two or three settings it was possible to set within one half division, corresponding to 0.5″ of arc or 0.02″ per centimeter length of specimen, for small angles. With larger deflections the image became less distinct so that the accuracy of setting was not quite as great but in all cases amply sufficient. Considerable difficulty in obtaining measurements was experienced due to vibrations in the building or produced by passing street cars, so that it was found necessary to take data at night when traffic was reduced.

A diagram of the electrical connections is shown in Fig. 1. A special form of rheostat, R_1, made of carbon lamps was selected, which although it did not have the ease of manipulation nor advantage of continuous change of resistance possessed by a liquid or slide-wire rheostat, had the very decided advantage of certainty of setting permitting readings to be easily repeated, which was very desirable in this work. The rheostat was so constructed that all the lamps could easily be placed in series, in parallel, or in various combinations of lamps in parallel with a group of lamps. For example, with switches *a*, *b* and *c* closed, lamp *1* was connected directly across the line in parallel with *3*, *4* and *5* in series. This arrangement made it possible to increase the current from zero to a maximum by as few or as many steps as desired. The adjustable rheostat, R_2, was connected in parallel with R_1 to obtain the higher values of current.

Current through the specimen was controlled by means of two field rheostats, R_3 and R_4, in parallel with a slide wire rheostat, R_5. The latter provided for fine adjustment necessary to keep the current constant during a run.

The current was read by means of Weston milli-voltmeters provided with one and five ampere shunts. They were frequently calibrated by means of a potentiometer, standard cell and standard resistances.

The solenoid was 38 cm. in length and hence the field was not quite uniform throughout the 25 cm. length of the specimen. From approximate dimensions the variation from the maximum, of the average field over the length of the specimen was computed. As no accurate data for the solenoid were obtainable it was necessary to determine its constant experimentally. This was done by means of a ballistic galvanometer and a ballistic coil whose dimensions were accurately known. The galvanometer was calibrated by means of a standard condenser. The constant after applying the correction indicated above was found to be 210.5 gauss per square cm. per ampere. A later determination using a mutual inductance instead of the condenser gave a result differing from that given above by only a small fraction of one per cent.

Method of Observation.

The specimen was thoroughly demagnetized before each run by gradually decreasing an alternating current flowing through the solenoid, which process was repeated several times with decreasing voltages applied. Demagnetization by less frequent reversals made by hand and also by a mechanical commutator constructed for the purpose, was also tried and as no difference was observed it was concluded that the demagnetization

by A.C. was complete in most cases. (Certain exceptions will be noted later.)

Keeping the circular field constant four runs were made as follows: both longitudinal and circular fields direct; both reversed; circular field direct, longitudinal field reversed; circular field reversed, longitudinal field direct. In many cases one or more repetitions of these runs were made. When both fields were direct the relation of circular and longitudinal fields and of twist for iron, was right handed or positive according to the convention previously mentioned.

DATA AND RESULTS.

Examination of the plotted results showed that these four runs gave two distinct curves, one when both fields were direct or reversed, the other when either one was reversed. Which one of the curves was obtained evidently depended upon the direction of twist in the specimen since the agreement between the curves of each pair seems to eliminate other possibilities. This lack of agreement has been noticed by other investigators and ascribed to various causes, consequently a study of this effect was made to determine its origin.

Curves *1* and *2*, Fig. 2, show typical results for cobalt specimen A, when both fields are direct and when one is reversed respectively. Curve *3* is the average of curves *1* and *2*.

Curves *5*, *6* and *7* show the same thing for iron specimen H; and curves *9*, *10* and *11* for nickel specimen E.

It was found in the case of every specimen that after careful demagnetization the application of either the longitudinal or circular field alone produced a small twist, whose value varied with different specimens and the magnitude of the magnetizing field. The initial twist due to the circular field was, however, extremely small for most of the specimens. The maximum value of the twist produced by the longitudinal field varied from 0.18″ to 4.33″ per cm. in different specimens and, although its value varied somewhat with different runs on the same specimen, it was always in the same direction and seemed to be practically independent of the previous condition of magnetization or of demagnetization so long as the latter was reasonably thorough. In general the twist was greater in unannealed than in annealed specimens of the same material. Curves *4*, *8* and *12* show average results for these correction curves for specimens A, H and E respectively.

In the case of nickel the initial twist due to the circular field varied quite erratically with different runs both in magnitude and direction. However, if the twist, as measured from the original zero when neither

field was operating, be plotted from the correction curve obtained when the longitudinal field alone was operating, as a new zero axis of twist the resulting corrected curves almost coincide. An example of the result of this operation is shown in Fig. 2 by the double row of dots, obtained

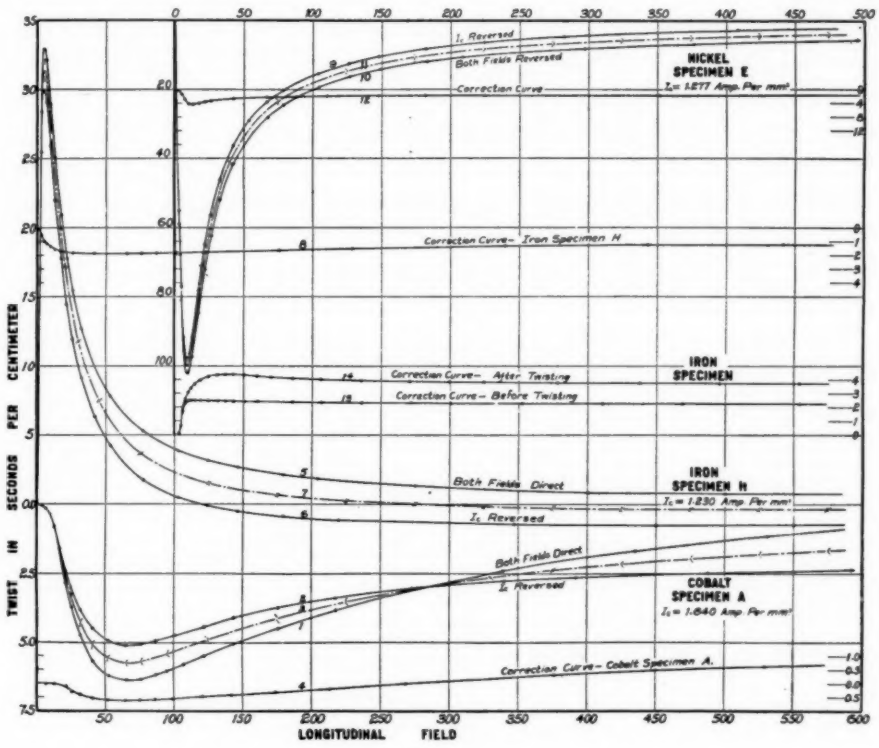

Fig. 2.

by applying curve *12* as a correction to curves *9* and *10*. It is seen the dots almost coincide with curve *11* which is the average of curves *9* and *10*. In some cases there was some divergence near the peaks of the curves but the agreement was always good for higher values of the longitudinal field.

Treating the data for iron in the same manner gave even better results, an example of which is shown by the dots almost coinciding with curve *7*, obtained by applying curve *8* as a correction to curves *5* and *6*.

In the case of cobalt, however, the best agreement was obtained by measuring the twist not from the initial zero but from that obtained after the application of the circular field.

An analysis of these rather complex results indicates that the lack of symmetry in twist is apparently due to various combinations of the three

following factors: imperfect demagnetization, change in temperature of the specimen due to the heating effect of the current in it, and æolotropic structure of the specimen.

Imperfect demagnetization was in evidence in the annealed nickel specimens only, which apparently accounts for the initial twist of erratic character due to circular field. This is not at all surprising in this case since for very small values of the longitudinal field, not only is the susceptibility relatively high making perfect demagnetization difficult, but the twist is also very large, thus greatly magnifying the effect of even an extremely small amount of residual magnetism. Indeed, it may be added that this method affords an extremely delicate means of detecting residual magnetism in annealed nickel. No such effect is observed in hard drawn nickel since both the initial susceptibility and the twist are very low for small values of the longitudinal field.

The fact that this initial twist in cobalt and iron due to the circular field alone, was always in the same direction and of about the same magnitude for any given specimen seems to indicate that if residual magnetism were present it was so small that its effect could not be observed. The observed twist was evidently due to heating and æolotropy in these cases.

Twist caused by either of these two effects alone could be distinguished by the fact that æolotropy produces a true magnetic twist for which correction is necessary as in the case of iron; while the effect of heating is merely to change the zero of twist by producing a shift in the equilibrium configuration of the specimen allowing a partial adjustment of residual strains. Such was the case with cobalt. However, if both of the above effects be present, the net twist due to circular field is the algebraic sum of the two components, correction for only one of which should be made. Consequently, the application of the corrections in the manner previously described results, in general, in two parallel curves, the intercept between which is twice the variation of the correction actually made for the circular field (it was zero in the case of cobalt) from the component due to æolotropy alone.

Since the corrected curves for iron and cobalt almost coincide as shown by the curves, we conclude that the twist due to circular field in the case of the former is due almost wholly to æolotropy, and to rise of temperature in the case of the latter.

That this initial effect in cobalt is due primarily to change in temperature is further supported by the following facts. Not only did the maximum twist show a very marked increase with rise of temperature, but in one case after making a run at approximately 60 degrees centi-

grade and demagnetizing, a twist of approximately 4.0″ per cm. was observed on reducing the temperature to approximately 15 degrees.

Furthermore, the initial twist in cobalt seemed to increase approximately as a parabolic function of the current which one would expect if due to the heating effect.

On the other hand the effect of temperature change in iron and nickel was so small as to be almost completely masked by the other factors.

The twist produced in all specimens by the application of the longitudinal field alone is apparently due almost wholly to æolotropy caused perhaps by permanent residual strains sustained in drawing, by subsequent coiling of the wire, etc. As we have seen, such an effect may give a circular component to longitudinal magnetism.

This view is supported by the fact that annealing reduced the effect, and by the following experiment. One end of an iron specimen which

TABLE II.

Annealed Iron Specimen H. Current, Ic, in Specimen in Amperes per Square Millimeters.

$I_c = 0.1229$.		$I_c = 0.2951$.		$I_c = 0.615$.		$I_c = 1.230$.	
H.	*θ.*	*H.*	*θ.*	*H.*	*θ.*	*H.*	*θ.*
1.50	0.87	1.50	2.50	1.39	8.11	2.93	26.90
2.14	2.32	2.14	6.02	2.14	14.41	4.71	31.39
3.21	3.28	3.21	8.21	3.16	18.12	6.64	30.61
4.28	3.72	4.28	9.16	4.28	19.52	9.33	27.43
5.57	3.74	5.57	9.18	5.56	19.29	12.74	23.49
7.28	3.57	7.28	8.71	7.33	18.01	17.08	19.34
9.21	3.29	9.31	7.96	9.26	16.45	21.41	16.08
11.25	3.01	11.30	7.30	11.30	14.90	25.9	13.61
14.35	2.66	14.35	6.41	14.35	12.94	32.3	10.89
18.42	2.26	18.41	5.43	18.40	10.82	53.3	6.01
23.88	1.83	23.88	4.42	23.92	8.65	77.2	3.49
36.4	1.18	30.4	3.41	30.4	6.76	102.7	2.16
49.5	0.79	49.7	1.89	36.4	5.57	124.5	1.51
69.3	0.50	59.3	1.45	49.7	3.75	146.3	1.06
84.6	0.30	69.4	1.14	59.3	2.92	179.3	0.61
104.8	0.23	84.7	0.81	69.4	2.30	204.1	0.38
133.8	0.15	104.9	0.54	84.7	1.71	275.5	0.00
168.7	0.08	133.9	0.30	104.9	1.19	347	−0.20
196.6	0.07	169	0.16	134.0	0.76	401	−0.30
246.5	0.02	197	0.10	169.1	0.48	450	−0.34
332	0.00	223	0.05	197.0	0.33	501	−0.35
362.5	−.003	275	−0.02	223	0.23	552	−0.34
450.2	−0.05	334	−0.06	274	0.03	608	−0.33
		392	−0.10	334	−0.03	659	−0.31
		452	−0.13	391	−0.10	708	−0.30
				451	−0.13	800	−0.25
						854	−0.22

$I_0 = 2.459$.		$I_0 = 3.554$.		$I_0 = 5.331$	
H.	θ.	H.	θ.	H.	θ.
2.89	24.35	2.5	18.37	2.3	12.04
4.6	33.98	4.1	27.71	4.1	20.32
6.6	37.68	5.6	34.55	5.7	28.31
9.4	37.12	7.7	38.85	7.8	35.42
12.8	33.80	10.4	39.62	10.5	39.50
17.1	29.07	14.0	37.27	14.1	40.52
21.5	24.82	17.4	33.92	17.8	39.23
26.1	21.40	21.1	30.25	21.5	36.17
32.6	17.42	26.5	25.80	27.1	31.95
42.1	13.41	34.6	20.62	35.1	26.35
53.4	10.11	44.1	16.01	44.9	20.87
77.7	6.08	54.2	12.61	66.0	13.53
103.0	3.86	65.7	9.95	88.1	8.98
125.3	2.67	73.0	8.59	106.3	6.76
147.4	1.87	83.0	6.39	126.1	4.86
179.7	1.11	106.2	4.68	154.0	3.01
205.7	0.71	126.0	3.36	190.9	1.55
282	−0.05	153.9	2.10	242.5	0.25
406	−0.59	190.7	1.09	300	−0.60
508	−0.68	242	0.24	353.5	−1.05
616	−0.59	329	−0.56	400	−1.33
715	−0.53	397	−0.92	445	−1.45
864	−0.49	490	−1.05	492	−1.45
		591	−0.99	544	−1.42
		685	−0.90	593	−1.37
		778	−0.80	637	−1.35
				700	−1.28
				779	−1.19

had been tested previously, was twisted through 360 degrees, leaving a permanent twist of 180 degrees. Subsequent tests showed that the average maximum twist was reduced and in strong fields the twist was increased in one direction and decreased in the other. The twist due to either circular or longitudinal field alone was greatly increased, the latter being almost doubled as shown by curves *13* and *14*, Fig. 2. Data for curve *13* were taken before and for curve *14* after twisting.

One peculiarity still remains unexplained. In some cases the corrected curves showed a considerable divergence in the region of maximum twist. It is perhaps due to a difference in the modulus of rigidity for direct and reversed twist, as magnetic twist must always be accompanied by actual mechanical strain since the tangential component of shear due to the magnetic fields does not vary uniformly from the center to the circumference of the wire.

At any rate it seems that one may safely conclude that the average curve for any given specimen gives very closely at least, the single curve

that would be obtained if experimental conditions could be better controlled and the specimen freed from accidental eccentricities.

In this work the data and curves shown are the result of selecting data from runs giving nearly average results for twist in the two directions and averaging these results. Where runs could not be found very closely approximating the average result an actual average of the different runs is given.

The results for iron are shown in Table II. and in Fig. 3 for several values of current as indicated. In general characteristics the results

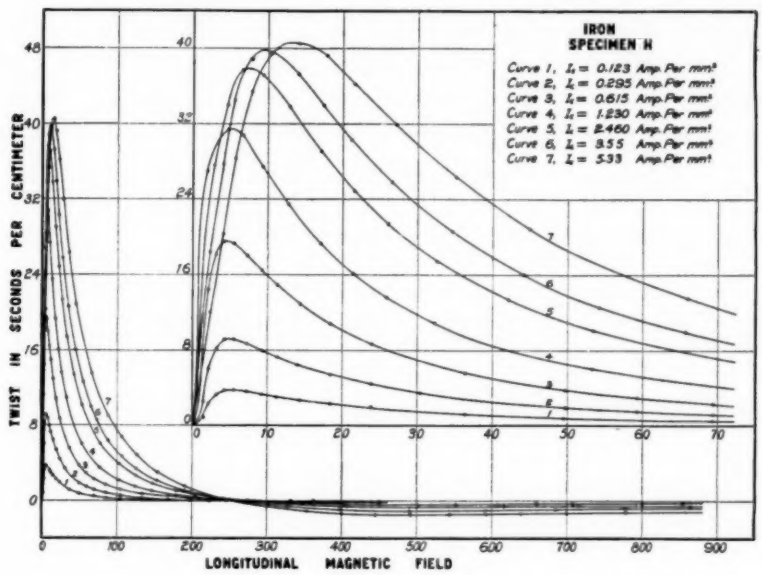
IRON
SPECIMEN H
Curve 1, I_1 = 0.123 Amp. Per mm²
Curve 2, I_2 = 0.295 Amp. Per mm²
Curve 3, I_3 = 0.615 Amp. Per mm²
Curve 4, I_4 = 1.230 Amp. Per mm²
Curve 5, I_5 = 2.460 Amp. Per mm²
Curve 6, I_6 = 3.55 Amp. Per mm²
Curve 7, I_7 = 5.33 Amp. Per mm²
TWIST IN SECONDS PER CENTIMETER
LONGITUDINAL MAGNETIC FIELD

Fig. 3.

agree quite well with those obtained by other observers; in fact they show a striking similarity to results obtained for steel tubes by S. R. Williams.[1] It will be observed that the values of the maximum twist increase roughly proportional to the current in the specimen at first then seemingly approach a maximum. This is what one would expect if the Wiedemann effect is a special case of the Joule effect, since the elongation in iron reaches a maximum in fairly low fields, and consequently with large circular fields the magnitude of the resultant field and its angle with the axis of the wire do not combine to form a condition so favorable for a large twist.

The maxima which gradually move into higher longitudinal fields with increase of circular field, are not only very sharp in this specimen but also occur at somewhat lower fields than observed by others. This is

[1] *Loc. cit.*

doubtless due to the extremely soft, well-annealed condition of the iron, since in another specimen evidently not so well annealed by alternating current, the maxima were broader and came in stronger fields.

The curves show rather striking characteristics in stronger fields. Not only do they all reverse their direction of twist at approximately the same value of longitudinal field,—a fact the reason for which is not apparent,—but in still higher fields they reach a maximum negative twist and at the highest values of field obtained, seem to be very gradually approaching the axis of zero twist. So far as the author knows this characteristic has never been previously observed. Apparently in only one other instance[1] has the Wiedemann effect been studied with values of longitudinal field above 400 or 500 gauss, and that was with rods about one centimeter in diameter whose characteristics were quite different. A further study of this feature,—for which the Joule effect offers no apparent explanation,—will be necessary to determine its cause and whether it is characteristic of all well annealed iron specimens.

The results for annealed nickel specimen E are given in Table III.

TABLE III.

Annealed Nickel Specimen E. Current, Ic, in Specimen in Amperes per Square Millimeters.

$I_c = 0.306$.		$I_c = 0.639$.		$I_c = 1.277$.	
H.	*θ.*	*H.*	*θ.*	*H.*	*θ.*
2.9	16.57	2.9	39.2	2.8	56.8
4.5	27.17	4.5	55.0	4.6	79.0
6.5	32.97	6.5	63.8	6.6	93.7
9.2	33.30	9.2	64.6	9.4	99.8
12.7	28.88	12.7	56.5	12.8	94.5
16.86	23.17	17.1	45.8	17.1	81.5
21.3	18.64	21.4	37.4	21.6	69.0
25.7	15.61	25.8	31.5	26.1	59.4
32.1	12.68	32.2	25.6	32.7	49.0
41.7	9.90	41.8	20.1	42.4	39.0
52.9	7.86	53.2	16.07	66.8	25.85
77.0	5.80	77.4	11.39	78.3	22.43
102.5	4.36	102.8	8.81	103.8	17.28
124.3	3.64	124.8	7.34	126.4	14.32
146.0	3.16	146.6	6.23	148.3	12.18
179.7	2.64	180.2	5.12	182.3	9.83
223.2	2.10	224	4.13	216.4	8.40
279	1.80	280	3.33	289.5	6.30
349	1.38	406	2.32	392	4.69
405	1.18	508	1.86	503	3.66
507	1.00	619	1.54	620	3.03
619	0.84	721	1.36	720	2.59
718	0.80	867	1.15	866	2.19
867	0.66				

[1] Honda and Shimizu, Phil. Mag., Vol. 5, S. 6, p. 650.

$I_c = 2.554$.		$I_c = 3.691$.		$I_c = 5.536$.	
H.	*θ.*	*H.*	*θ.*	*H.*	*θ.*
2.7	62.1	2.9	62.9	2.9	56.6
4.5	90.3	4.6	93.4	4.50	88.8
6.4	113.2	6.5	119.5	6.4	116.9
8.9	130.7	9.3	142.4	9.1	146.7
12.4	137.2	12.7	155.4	12.5	170.5
16.5	132.3	17.0	157.7	16.3	186.2
20.8	121.4	21.5	149.7	21.2	189.9
25.1	109.7	25.9	139.2	25.6	186.3
31.3	94.8	32.3	123.5	32.0	174.8
40.5	77.8	41.8	104.0	41.3	154.8
51.5	63.9	53.1	86.4	52.4	132.4
65.4	51.9	67.1	71.2	76.3	98.0
75.0	46.1	77.2	62.8	101.1	75.7
102.0	34.6	102.4	48.2	122.8	63.0
120.8	29.56	124.3	40.1	144.9	53.8
142.0	25.09	146.1	33.9	174.6	44.39
174.1	20.51	179.6	27.7	220.5	35.30
206.47	17.31	222.7	22.40	275.4	28.35
271	13.16	280	17.77	344	22.72
338	10.54	348	14.28	396	19.60
392	9.06	403	12.36	495	15.67
489	7.28	502	9.86	598	13.07
593	6.03	607	8.29	694	11.40
688	5.22	701	7.08	833	9.40
824	4.33	838	5.95		

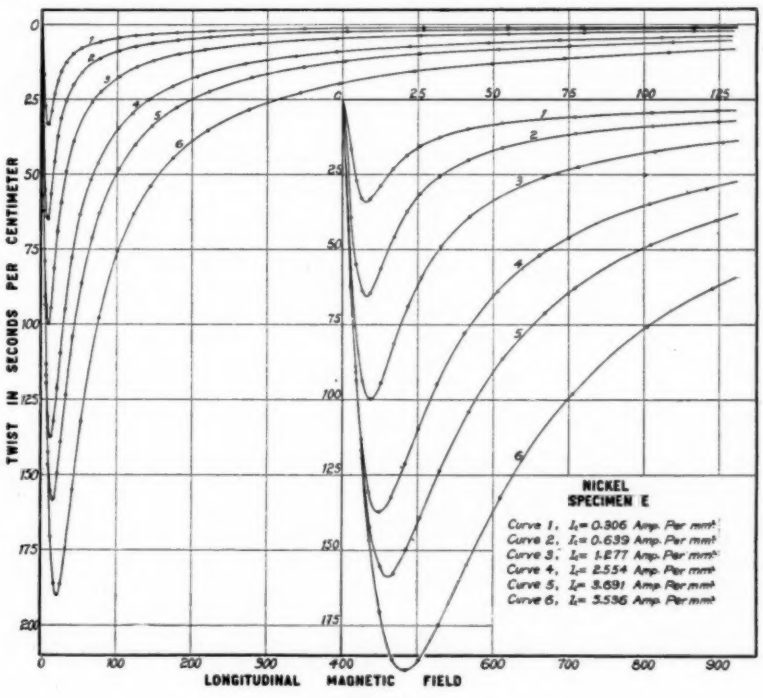
TWIST IN SECONDS PER CENTIMETER
LONGITUDINAL MAGNETIC FIELD
NICKEL
SPECIMEN E
Curve 1, I_c = 0.306 Amp. Per mm²
Curve 2, I_c = 0.639 Amp. Per mm²
Curve 3, I_c = 1.277 Amp. Per mm²
Curve 4, I_c = 2.554 Amp. Per mm²
Curve 5, I_c = 3.691 Amp. Per mm²
Curve 6, I_c = 5.536 Amp. Per mm²

Fig. 4.

TABLE IV.

Unannealed Nickel Specimen F. Current, Ic, in Specimen in Amperes per Square Millimeters.

$I_c = 0.641$.		$I_c = 1.281$.		$I_c = 2.563$.		$I_c = 3.704$.	
H.	*θ.*	*H.*	*θ.*	*H.*	*θ.*	*H.*	*θ.*
2.7	0.02	2.6	0.02	2.5	0.09	2.7	0.14
4.5	0.05	4.3	0.06	4.3	0.17	4.5	0.235
6.4	0.065	6.3	0.09	6.2	0.24	6.4	0.36
12.6	0.14	8.6	0.15	8.6	0.36	9.1	0.51
21.4	0.32	11.8	0.21	11.8	0.49	16.9	1.10
32.2	0.73	15.7	0.34	19.9	1.06	25.8	2.74
41.8	1.08	19.9	0.50	30.1	2.66	41.7	6.54
53.2	1.44	29.9	1.18	38.9	4.19	53.3	8.30
66.2	1.74	38.7	1.98	49.4	5.50	65.9	9.66
77.3	1.92	49.2	2.68	61.4	6.54	77.2	10.62
88.2	2.03	61.2	3.23	71.9	7.21	102.5	11.63
102.6	2.12	82.0	3.83	82.3	7.68	117.6	11.96
115.8	2.16	95.2	4.06	95.9	8.08	124.2	12.04
124.5	2.17	101.5	4.14	101.7	8.24	136.4	12.07
136.9	2.19	109.0	4.22	109.3	8.37	146.0	12.06
146.5	2.19	115.0	4.26	115.9	8.45	163.9	11.96
164.2	2.17	126.4	4.29	127.4	8.54	179.2	11.78
179.9	2.13	135.1	4.30	136.0	8.56	204.0	11.46
204.5	2.06	151.0	4.28	152.0	8.52	222.2	11.16
223	2.01	165.1	4.22	166.4	8.44	278	10.26
256	1.92	187.6	4.14	188.6	8.26	347	9.18
348	1.65	203.5	4.06	205.5	8.08	400	8.49
404	1.52	232	3.90	237	7.72	450	7.91
505	1.32	252	3.80	257.5	7.48	499	7.40
613	1.16	285	3.60	295	7.07	555	6.90
711	1.04	314	3.42	321	6.80	603	6.51
785	0.98	363	3.18	373	6.28	655	6.12
851	0.92	407	2.96	417	5.88	695	5.84
		455	2.78	464	5.50	760	5.44
		498	2.60	508	5.18	825	5.06
		550	2.44	565	4.82		
		595	2.31	609	4.56		
		640	2.19	660	4.28		
		682	2.08	695	4.12		
		682	2.08	777	3.78		
		765	1.90				

and Fig. 4. They agree remarkably well with results obtained by other observers. It will be observed that for nickel the twist is opposite in direction to that in iron, does not reverse its direction and the maximum value is more than four and one half times as great as in iron. The maximum twist in iron for the highest value of current used is approximately 40.5″ per cm., while for a slightly higher value of current in nickel it is 190″ per cm. As in iron the peaks advance at first slowly, then more

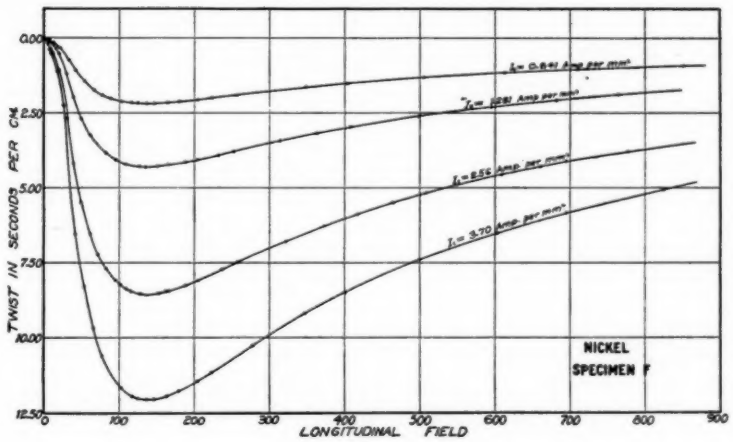

Fig. 5.

TABLE V.

Annealed Cobalt Specimen A. Current, Ic, in Specimen in Amperes per Square Millimeters.

$I_c = 0.393$.		$I_c = 0.820$.		$I_c = 1.640$.	
H.	*θ.*	*H.*	*θ.*	*H.*	*θ.*
4.3	0.01	4.5	0.06	2.7	0.04
6.4	0.02	6.4	0.10	4.5	0.08
9.0	0.04	9.1	0.22	6.5	0.17
12.6	0.18	12.6	0.41	9.2	0.34
16.9	0.37	16.9	0.83	12.6	0.82
21.2	0.59	21.2	1.29	16.9	1.62
25.6	0.76	25.5	1.69	21.2	2.63
32.0	0.97	31.9	2.13	25.6	3.48
41.6	1.19	41.3	2.55	32.0	4.27
53.0	1.31	52.5	2.84	41.4	5.09
65.9	1.33	65.0	2.94	52.9	5.57
74.5	1.31	76.1	2.92	65.3	5.72
87.2	1.29	86.7	2.86	76.4	5.69
101.5	1.22	100.3	2.77	87.0	5.55
123.1	1.14	121.8	2.58	101.0	5.35
144.7	1.05	143.0	2.39	122.4	4.96
177.5	0.95	175.6	2.18	143.8	4.57
201.9	0.87	216.5	1.91	176.0	4.11
275	0.65	272.5	1.66	217.5	3.62
341	0.55	339	1.41	273.0	3.10
394	0.49	391	1.30	340	2.59
490	0.42	486	1.10	392	2.30
594	0.33	533	1.01	488	1.95
686	0.26	679	0.85	592	1.64
819	0 20	811	0.75	685	1.42
				825	1.13

$I_c = 3.279$.		$I_c = 4.734$.		$I_c = 7.102$.	
H.	θ.	H.	θ.	H.	θ.
2.7	0.10	2.5	0.24	2.5	0.72
4.6	0.20	4.3	0.52	4.3	1.48
6.5	0.43	6.4	0.98	6.4	2.66
9.1	0.95	8.9	1.97	12.5	7.77
12.7	2.03	12.5	3.79	20.1	15.45
16.9	3.92	16.9	6.41	25.5	18.99
21.2	5.80	21.2	9.07	31.8	22.58
25.6	7.25	25.6	11.27	41.2	25.83
32.1	9.06	32.0	13.77	52.3	27.48
41.6	10.63	41.6	15.96	64.8	27.69
52.9	11.55	53.0	17.10	75.8	27.17
65.8	11.83	65.7	17.38	86.5	26.40
77.6	11.70	76.6	17.14	100.5	25.19
87.1	11.46	87.4	16.70	122.1	23.25
95.5	11.19	101.7	16.00	143.8	21.45
101.5	11.00	123.4	14.84	176.4	18.98
123.3	10.20	145.0	13.67	200.5	17.47
144.5	9.45	178.1	12.18	275	13.94
177.5	8.48	202.8	11.26	341	11.76
219.8	7.40	276	9.04	393	10.45
276	6.36	344	7.66	489	8.61
342	5.37	399	6.82	590	7.30
396	4.77	498	5.66	680	6.37
494	4.00	603	4.83	807	5.42
596	3.40	695	4.24		
687	3.00	827	3.57	8.9	4.78
817	2.55			16.8	11.93

rapidly into higher longitudinal fields with increasing circular field. However, the peaks of the curves seem to approach a maximum value of twist much less rapidly than in the case of iron. Here again the sharply defined maxima occurring in comparatively low fields are an indication of good annealing. A specimen of the same wire not so well annealed by alternating current gave curves (not shown) having flatter maxima of less magnitude and occurring in higher longitudinal fields as in the case of iron.

This is very strikingly illustrated by the results for specimen F, shown in Table IV. and Fig. 5, which was a hard drawn nickel wire exactly the same as specimen E except that it was not annealed. Here the broad flat maxima, not only occur in very much higher fields, but have only about one thirteenth the magnitude of the former. It clearly shows the importance in work on magneto-striction of knowing the previous history of the specimen, especially the heat treatment.

In Table V. and Fig. 6 are shown the results for cobalt specimen, A, and in Table VI. and Fig. 7 for the less pure specimen, C. A comparison

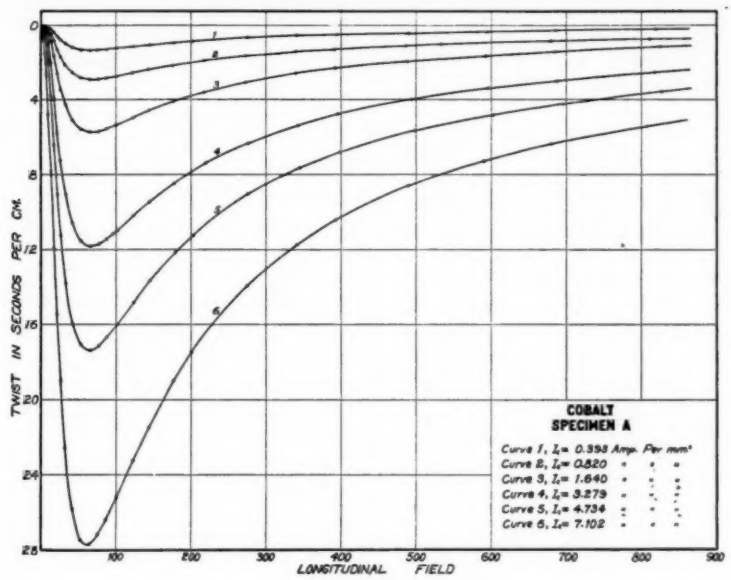

Fig. 6.

TABLE VI.

Annealed Cobalt Specimen C. Current, Ic, in Specimen in Amperes per Square Millimeters.

$I_c = 0.308$.		$I_c = 0.771$.		$I_c = 1.542$.	
H.	*θ.*	*H.*	*θ.*	*H.*	*θ.*
2.7	0.02	2.7	0.04	2.7	0.04
4.3	0.03	4.3	0.06	4.3	0.10
6.4	0.05	6.4	0.09	6.4	0.17
9.0	0.06	9.0	0.16	9.0	0.31
12.5	0.12	12.54	0.32	12.5	0.65
16.8	0.35	16.90	0.92	16.8	1.92
21.1	0.75	21.2	1.90	21.2	4.01
25.5	1.13	25.5	2.83	25.5	5.78
31.8	1.51	31.8	3.77	31.8	7.61
41.2	1.77	41.2	4.40	41.3	8.92
52.3	1.84	52.4	4.62	52.6	9.22
65.0	1.80	65.1	4.52	65.3	9.02
76.1	1.74	76.3	4.34	76.4	8.63
86.9	1.62	87.0	4.12	87.2	8.21
100.9	1.54	101.2	3.83	101.2	7.66
122.6	1.37	123.1	3.43	123.1	6.84
144.2	1.24	144.7	3.06	144.7	6.14
177.2	1.07	177.5	2.62	177.7	5.28
201.5	0.94	202.1	2.37	202.2	4.74
275.7	0.73	276.2	1.83	276.2	3.60
343.6	0.57	345	1.47	346	2.92
399	0.49	400	1.27	399	2.54
499	0.41	500	1.01	498	2.03
608	0.32	608	0.84	608	1.65
705	0.28	704	0.71	703	1.41
852	0.23	850	0.59	847	1.17

$I_c = 3.083$.		$I_c = 4.451$.		$I_c = 6.677$.	
H.	θ.	H.	θ.	H.	θ.
2.7	0.12	2.7	0.27	2.7	1.20
4.5	0.23	4.5	0.53	4.5	2.52
6.4	0.42	6.4	0.94	6.4	4.66
9.1	0.80	9.1	1.83	9.08	9.00
12.6	1.78	12.6	4.23	13.5	16.32
16.9	5.00	16.96	10.20	16.9	26.55
21.2	9.33	21.2	16.58	21.2	35.60
25.6	12.93	25.6	21.40	25.5	42.25
31.9	16.68	32.0	25.85	31.85	48.42
41 5	18.53	41.6	28.69	41.3	51.88
52.7	19.07	52.9	29.13	52.4	51.68
65.5	18.57	65.6	28.08	65.1	49.10
76.5	17.72	76.5	26.67	76.3	46.10
97.3	16.81	87.2	25.25	87.0	43.18
101.6	15.62	101.7	23.32	101.1	39.40
123.2	13.93	123.3	20.73	122.9	34.49
144.7	12.46	144.8	18.41	144.4	30.15
177.8	10.64	177.9	15.65	177.1	25.23
210.8	9.27	202.2	14.01	201.6	22.34
276	7.25	276.1	10.51	275.8	16.28
341	5.89	344	8.49	343	12.83
397	5.06	398	7.29	398	10.92
497	4.01	496	5.78	498	8.46
602	3.31	601	4.73	604	6.78
695	2.82	694	4.01	696	5.71
832	2.32	828	3.28	833	4.60

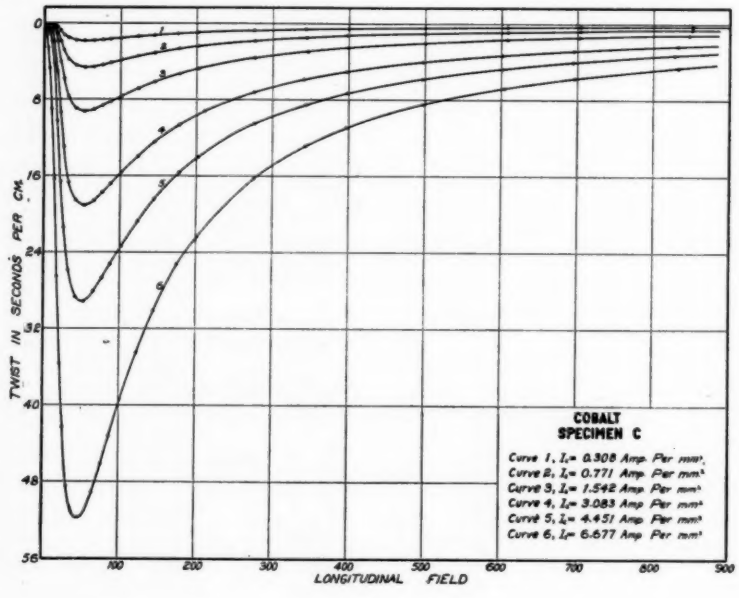

Fig 7.

of the curves shows that the maximum twist for the purer specimen is much less than for the other and occurs at higher values of the longitudinal field. This is not surprising when we consider that the purer cobalt is more highly crystalline and consequently exhibits a greater magnetic hardness and a correspondingly lower susceptibility; properties which as we have seen in the case of nickel, produce exactly this sort of an effect.

These results for pure cobalt differ very materially from those obtained

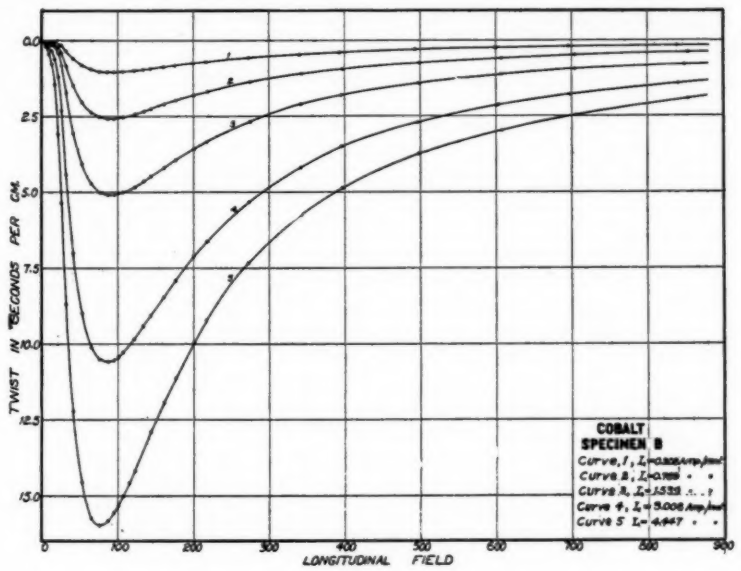

Fig. 8.

by Honda,[1] for not only is there no reversal of twist in strong fields such as he found for a cast cobalt rod, but the curves show very little resemblance to those he obtained for an annealed cobalt rod. The difference is doubtless due to the difference in purity of cobalt.

Examination of the curves also shows that there is a very marked similarity between the results for cobalt and nickel, making due allowance for the difference in the magnitude of twist in the two cases. In fact, there is actually less difference between the curves for annealed specimens of cobalt and nickel than between those for annealed and unannealed nickel. (It will be shown in a later publication that this similarity holds also for the Joule effect.) This suggests the possibility that, if a more satisfactory method of annealing cobalt could be found reducing it to a less marked crystalline condition or at least to a finer crystalline texture,

[1] Honda and Shimiza, Phil. Mag., Vol. 5, S. 6, p. 650.

the results might compare even more favorably with those for nickel. Such a suggestion receives some support from the results for cobalt specimen B, given in Table VII. and Fig. 8. This specimen differs from specimen C, only in not being subjected to further annealing.

Here again we see the marked effect of annealing and that the result corresponds exactly with that found in iron and nickel. Indeed it may be stated as a general characteristic of all three metals that, the more thorough the annealing the greater is the magnitude of the maximum twist, the sharper the peaks of the curves, and the lower the values of the longitudinal field in which they occur. Now this is exactly the sort of effect that annealing has upon the susceptibility. So, qualitatively

TABLE VII.

Unannealed Cobalt Specimen B. Current, Ic, in Specimen in Amperes per Square Millimeters

$I_c = 0.308$.		$I_c = 0.769$.		$I_c = 1.539$.	
H.	*θ.*	*H.*	*θ.*	*H.*	*θ.*
4.3	0.008	2.6.	0.01	2.6	0.02
9.0	0.02	4.4	0.03	4.5	0.04
16.6	0.04	6.4	0.04	6.4	0.06
20.9	0.07	9.0	0.05	9.0	0.08
25.4	0.15	12.4	0.06	12.5	0.13
31.7	0.34	16.7	0.11	16.7	0.21
40.8	0.58	21.0	0.20	21.1	0.40
52.0	0.80	25.4	0.40	25.5	0.82
64.5	0.94	31.8	0.84	31.8	1.74
75.7	1.02	40.97	1.49	40.9	3.05
86.2	1.03	52.1	2.04	52.2	4.07
94.5	1.04	64.7	2 37	64.7	4.71
100.2	1.02	75.9	2.52	76.1	5.01
106.8	1.02	86.5	2.57	86.6	5.09
121.7	0.98	94.8	2.57	95.0	5.08
133.7	0.95	100.5	2.55	100.6	5.05
143.0	0.92	107.1	2.52	107.1	5.00
160.4	0.86	122.1	2.44	122.2	4.84
175.5	0.81	134.1	2.34	134.1	4.64
199.7	0.74	143.7	2.26	145.6	4.49
216.4	0.69	161.2	2.11	161.0	4.20
272.5	0.56	176.2	1.98	176.5	3.95
340	0.45	200.6	1.81	200.6	3.58
393	0.39	218.7	1.69	217.9	3.34
493	0.28	274.5	1.36	275	2.69
599	0.24	343	1.10	342	2.12
695	0.20	397	0.93	395	1.82
839	0.16	498	0.73	497	1.41
		607	0.58	606	1.12
		704	0.47	704	0.93
		853	0.38	849	0.78

$I_c = 3.008$.		$I_c = 4.447$.		$I_c = 3.008$.		$I_c = 4.447$.	
H.	θ.	H.	θ.	H.	θ.	H.	θ.
2.6	0.05	2.6	0.10	106.9	10.27	107.1	15.00
4.4	0.08	4.3	0.18	121.8	9.83	115.3	14.57
6.4	0.13	6.4	0.26	133.8	9.42	122.2	14.17
9.00	0.22	9.0	0.44	143.2	9.07	143.7	12.90
12.4	0.35	12.5	0.75	160.7	8.45	161.2	11.93
16.8	0.61	16.7	1.46	176.0	7.90	176.5	11.12
21.0	1.20	21.0	3.08	200.2	7.11	200.9	9.96
25.5	2.30	25.4	5.35	217.9	6.64	218.7	9.22
31.7	4.42	31.7	8.68	273.6	5.34	275	7.34
40.9	7.02	40.97	12.21	342	4.20	343	5.75
52.1	8.99	52.1	14.55	396	3.50	397	4.87
64.7	10.11	64.7	15.73	495	2.72	498	3.74
75.7	10.50	76.0	15.98	602	2.15	608	2.97
86.5	10.59	86.6	15.82	697	1.79	705	2.46
94.6	10.54	95.0	15.55	841	1.40	854	1.93
100.4	10.42	100.7	15.31				

at least, there seems to be a close connection between the susceptibility and the Wiedemann effect. However, the latter is so complex that the relation is far from being a simple one. It is hoped that further study may bring out more definite relations.

The results for iron specimen H, nickel specimen E, and cobalt speci-

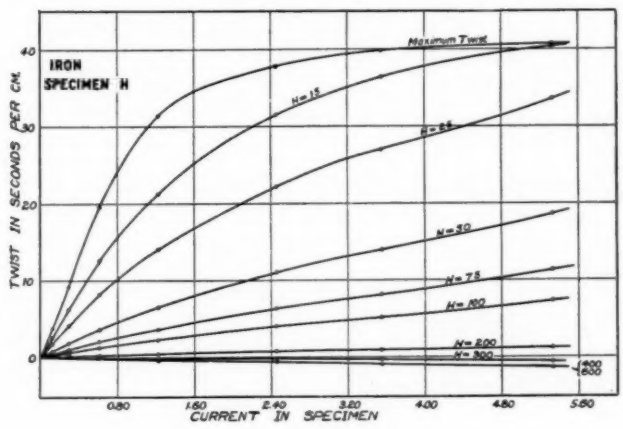

Fig. 9.

men C are presented in a somewhat different way in Figs. 9, 10 and 11 respectively, where for several values of the longitudinal field, values of current in the specimen are plotted as abscissæ and the corresponding twist as ordinates. In the case of nickel and cobalt it is seen that for values of the longitudinal field very large compared with the circular

field, the twist is approximately proportional to the latter. This is exactly what would be expected if the Wiedemann effect be a modification of the Joule effect. For, let P, Fig. 12, be any particle in a cylin-

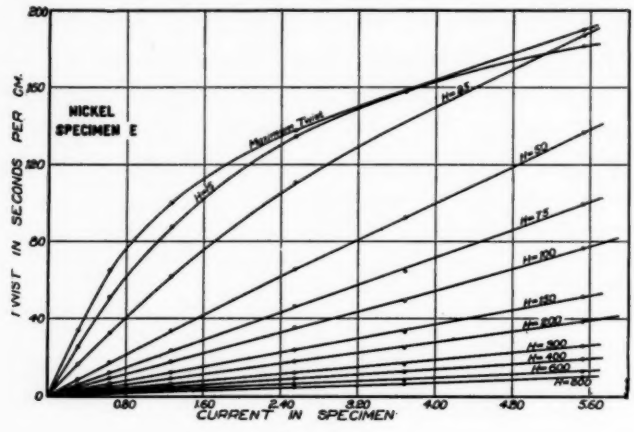

Fig. 10.

drical element of the wire. The approximate net effect of combined longitudinal and circular magnetization is to displace the particle through the distance λ, in the direction PP' of the resultant magnetic field, making

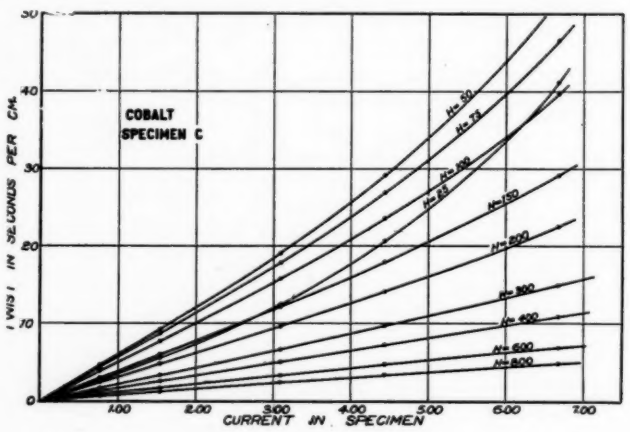

Fig. 11.

the angle θ with the length of the element in the direction of the axis of the wire. The twist, θ, is given by,

$$\theta = \frac{\lambda}{r} \sin \theta$$

$$= \frac{\lambda}{r} \frac{H_c}{\sqrt{H^2 + H_c^2}}$$

where H_c is the circular field, H the longitudinal field and r the distance of the particle from the axis. If H is very large compared to H_c then

$$\theta = \frac{\lambda}{rH} H_c.$$

Previous study of the Joule effect has shown that λ in nickel, reaches practically a constant value in moderately strong magnetic fields. From the equation it is seen that for constant values of H, θ is very nearly a linear function of H_c. This result, derived only for high values of magnetic field in a cylindrical element, at least gives an indication of what may be expected in a wire. For any given value of H_c the twist varies inversely as H, if H is very large, and consequently approaches zero as a

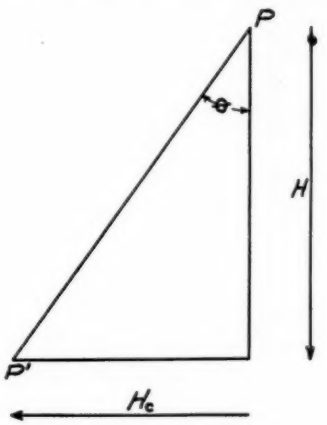

Fig. 12.

limit as H is increased indefinitely. The curves indicate that this is perhaps the case for nickel and cobalt.

These results correspond quite well with what one might predict from the molecular theory of magnetism; for in strong longitudinal fields which magnetize the material almost to saturation, the molecular magnets are almost all aligned in the direction of the field so that the only effect of the circular field is to produce a deflection of these molecular magnets proportional to the deflecting field with a corresponding deformation of the magnetized material.

Cobalt becomes saturated only in much stronger fields than nickel and consequently the agreement is not so good.

In this case as in that of all the magneto-elastic effects in iron, the results are very complex and any attempt to explain them without further study is fruitless.

Any analysis of magneto-elastic effects is made much more difficult by

the marked hysteresis occurring in all of them. In Figs. 13, 14 and 15 are shown hysteresis curves for iron, nickel and cobalt respectively, obtained by keeping the circular field constant and cyclically varying the longitudinal field. The comparatively large area inclosed in the hys-

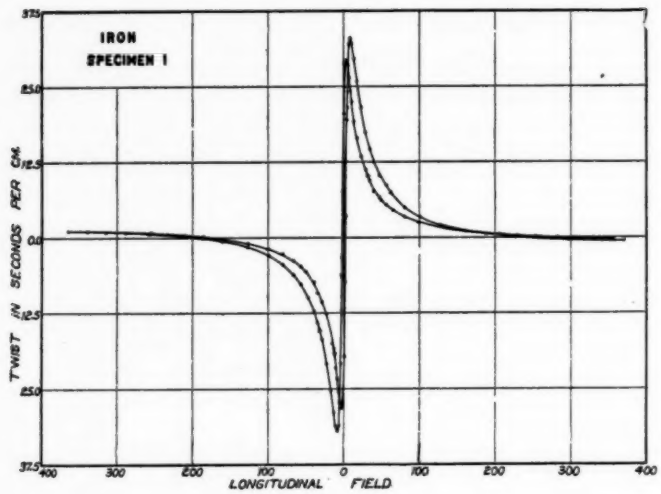

Fig. 13.

teresis loop for cobalt indicates its rather extreme magnetic hardness and low susceptibility, which has been previously mentioned.

Some of the features of the Wiedemann effect will be still further dis-

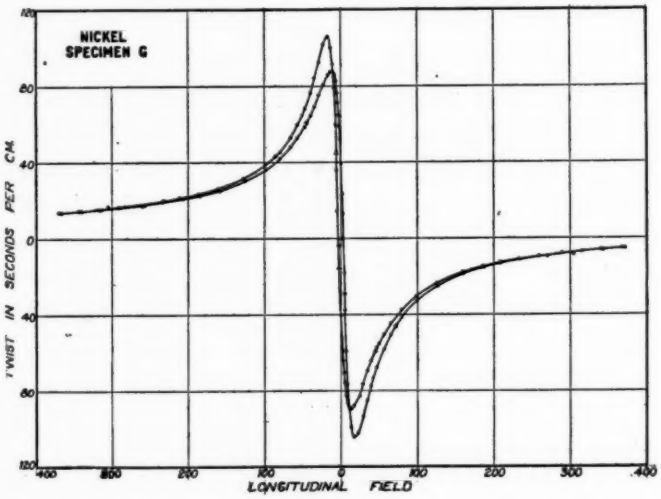

Fig. 14.

cussed in a later publication in connection with the Joule effect and other magnetic phenomena.

Summary.

Experimental curves are shown for the Wiedemann effect in specimens of pure cobalt wire, and for the purpose of comparison, results are also shown for specimens of iron and nickel subjected to the same heat treatment. A comparison of the results is made and certain features of the Wiedemann effect discussed.

The results for pure cobalt differ materially from those obtained by

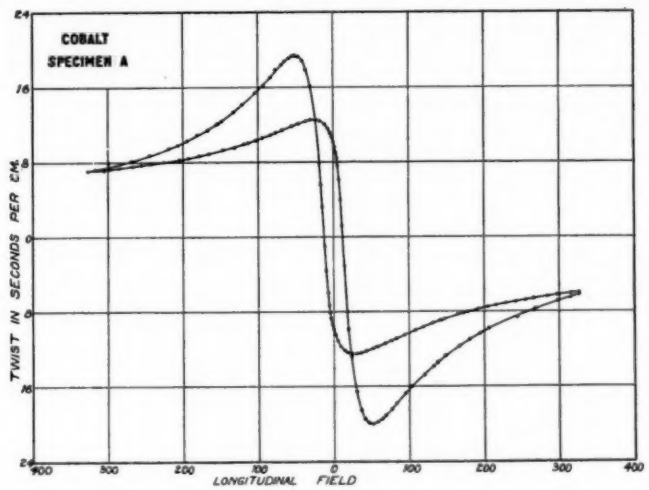

Fig. 15.

other observers working with impure specimens, and show a close similarity to the results for nickel, the twist, however being much less.

Certain eccentricities of twist observed in all specimens are studied and an explanation given which seems to account for nearly all of the observed facts.

Hysteresis curves for iron, nickel and cobalt are shown.

Experimental results for the Joule effect and other magnetic phenomena will be presented in a later publication.

The writer wishes to express his thanks to various members of the department staff for helpful suggestions given, and especially to Professor E. L. Nichols through whom the specimens were obtained and under whose direction the work was done.

Physical Laboratory,
Cornell University,
September, 1918.

ON THE MECHANICAL AND ELECTRODYNAMICAL PROPERTIES OF THE ELECTRON: ADDENDUM.

BY MEGH NAD SAHA.

IT appears that the relation

$$-\mu = -\frac{m_0c^2}{e} = \frac{f_{12}}{\Omega_{12}} = \frac{f_{23}}{\Omega_{23}} = \frac{f_{31}}{\Omega_{31}} = \frac{f_{14}}{\Omega_{14}} = \frac{f_{24}}{\Omega_{24}} = \frac{f_{34}}{\Omega_{34}},$$

which was given in my paper[1] "On the Mechanical and Electrodynamical Properties of the Electron" and which was obtained from the determinant

$$\begin{vmatrix} & f_{12}+\mu\Omega_{12} & f_{13}+\mu\Omega_{13} & f_{14}+\mu\Omega_{14} \\ f_{21}+\mu\Omega_{21} & & f_{23}+\mu\Omega_{23} & f_{24}+\mu\Omega_{24} \\ f_{31}+\mu\Omega_{31} & f_{32}+\mu\Omega_{32} & & f_{34}+\mu\Omega_{34} \\ f_{41}+\mu\Omega_{41} & f_{42}+\mu\Omega_{42} & f_{43}+\mu\Omega_{43} & \end{vmatrix} = 0$$

or

$$(f_{12}+\mu\Omega_{12})(f_{34}+\mu\Omega_{34}) + (f_{23}+\mu\Omega_{23})(f_{14}+\mu\Omega_{14}) + (f_{31}+\mu\Omega_{31})(f_{24}+\mu\Omega_{24}) = 0,$$

cannot hold. For, from the principle of least action written in Minkowski's form,

$$\delta\int m_0c^2ds - \int \delta W\cdot ds = 0,$$

we obtain the relation

$$\int\int[(f_{12}+\mu\Omega_{12})(dx\delta y - \delta x dy)] + 5 \text{ other similar terms} = 0.$$

But it is not possible to equate to zero the coefficients of the six-components $(dx\delta y - \delta x dy)$ of the area-six-vector $(dSx\delta s)$ as was done in that paper, for though $(\delta x, \delta y, \delta z, \delta l)$ represent an arbitrary displacement (dx, dy, dz, dl), is not so, but represent the actual displacements. We have, therefore, to collect the coefficients of $(\delta x, \delta y, \delta z, \delta l)$ and put them separately equal to zero. In this way we obtain

* PHYSICAL REVIEW, January, 1919.

$$-\frac{m_0c^2}{e}=\frac{f_{12}\omega_2+f_{13}\omega_3+f_{14}\omega_4}{\Omega_{12}\omega_2+\Omega_{13}\omega_3+\Omega_{14}\omega_4}=\frac{f_{21}\omega_1+f_{23}\omega_3+f_{24}\omega_4}{\Omega_{21}\omega_1+\Omega_{23}\omega_3+\Omega_{24}\omega_4}$$

$$=\frac{f_{31}\omega_1+f_{32}\omega_2+f_{34}\omega_4}{\Omega_{31}\omega_1+\Omega_{32}\omega_2+\Omega_{34}\omega_4}=\frac{f_{41}\omega_1+f_{42}\omega_2+f_{43}\omega_3}{\Omega_{41}\omega_1+\Omega_{42}\omega_2+\Omega_{43}\omega_3}, \quad (A)$$

which are simply another form of the Minkowskian equations

$$m_0c^2\frac{d^2x}{ds^2}=X, \quad m_0c^2\frac{d^2y}{ds^2}=Y, \quad m_0c^2\frac{d^2z}{ds^2}=Z, \quad m_0c^2\frac{d^2l}{ds^2}=L \quad (A')$$

for

$$\frac{d^2x}{ds^2}=-(\omega_2\Omega_{12}+\omega_3\Omega_{13}+\omega_4\Omega_{14}), \quad \text{etc.}$$

The form (A) as it involves the acceleration six-vector may for certain purposes prove more convenient than the form (A').

NEW BOOKS.

The Theory of the Relativity of Motion. By RICHARD C. TOLMAN. Berkeley: University of California Press, 1917. Pp. ix + 225.

This is an introduction to Einstein's physical theory of relativity and its developments, of a scope about like that of Cunningham's book.

The first chapter gives a historical review, a mere sketch but enough to hint at the development of the physical setting from Galileo and Newton to Michelson and Morley. Chapters two to six give an elementary account of the new kinematics and dynamics of a particle together with the optical matters of aberration, Doppler effect, and Fresnel convection. The material here is about that of Einstein's pioneer papers with one important and undoubtedly wise change, in that the introduction of the concept of mass in connection with phenomena of impact gives direct access to dynamics without requiring reference to particles electrically charged, though the field of a moving charge and the relations of mass and energy are reached as corollaries. This earlier portion of the book, nearly half, uses only elementary mathematical notions.

The rest of the volume is naturally less elementary. Chapters seven to eleven deal with dynamics of systems, some aspects of thermodynamics and statistical mechanics, and elastic bodies. In each case the principle of least action and Lagrange's equations have a fundamental place, in the last case with a frank avowal of the questions left open because of the lack of unique determination of the kinetic potential. Chapter twelve outlines the theory of the electromagnetic field, including brief reference to moving dielectrics. Here the whole development is based on an assumed form for the electromagnetic kinetic potential, giving an elegance of treatment that one would not like to miss. It may be asked however whether it would not have been more appealing to most readers to exhibit first, as Einstein did, the invariance of the Maxwell-Lorentz equations, and then pass over to the variational deduction, giving an order of thought more like that used for the dynamics. The final chapter is an outline of vector analysis in four dimensions, with a few examples of its use in formulating the theory previously given.

The book deserves hearty commendation as an interesting introduction to a subject often considered difficult of access, and will win the reader's thanks after he has pruned away a few obvious blemishes. Some passing remarks are distinctly irritating; for example, the reference to physical "model-making," one of the glories of physics, especially in the nineteenth century, as a "debauch." The reviewer was once reminded however by a biological friend that irritation and stimulation are not sharply distinct.

A. C. L.